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Some refinement of the notion of symmetry for the Volterra integral equations and the construction of symmetrical methods to solve them

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ABSTRACT

The theory of integral calculations is employed in most fields of the natural sciences for computing the volumes of rotating bodies, areas with different shapes, distances between objects, and other applications. They can be used to explore energy signals to study earthquakes, the distribution of telecommunications signals, and other uses. In particular, many problems formulated as mathematical models use integral equations with variable boundaries. The most interesting applications use integrals where the boundaries are symmetric functions. In the present study, to solve the Volterra integral equations with variable boundaries, we propose multi-step methods such as hybrids, and we construct some simple algorithms for their application where the order of accuracy is $p \leqslant 8$ $(k = 1)$.

1. Introduction

Many symmetrical objects exist in the natural world. In general, symmetrical objects exist for longer than unsymmetrical ones. Thus, people have understood the concept of symmetry for a long time. Therefore, many scientists consider that the issue of notions regarding the symmetry of some objects is resolved. However, we show that this assumption is not always valid.

The aim of this study is to extend the concept of symmetry to the Volterra integral equations and to construct methods using these new characteristics. According to previous studies, we can find a definition for a symmetric Fredholm integral equation (e.g., [1, p. 124]). We mainly use inductive methods to solve the problems described above, but it may be necessary to use deductive methods in some cases.

Consider the following nonlinear integral equation of Volterra type (e.g., see [1–16]):

$$y(x) = f(x) + \int_{x_0}^{x} k(x, s, y(s))ds, \quad x \in [x_0, X]. \tag{1}$$

There is a notion of the definition of symmetry for the following integral equation:

$$y(x) = f(x) + \int_{a}^{b} b(x, s)y(s)ds, \quad x \in [a, b]. \tag{2}$$

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This is a linear integral equation of Fredholm type. The symmetry of the integral equation (2) is determined by the symmetry of the kernel $b(x, s)$ of the integral. Thus, we need to use a different scheme in the nonlinear case, e.g.,

$$b(x, s)y(s) \equiv a(x, s, y(s)).$$

In this case, to define the symmetry of the integral equation (2), we can use the symmetry of the function $a(x, s, z)$.

Obviously, the integrals involved in Eqs. (1) and (2) are functions of the variable x; therefore, we can write the following

$$v(x) = \int_{x_0}^{x} k(x, s, y(s))ds.$$

The symmetry of the integral (1) can be defined by using $v(x)$ in the following form.

Definition 1. If the functions $v(x)$ and $f(x)$ are symmetrical, then Eq. (1) is called a symmetric integral equation of Volterra type.

We consider that there is no uniform definition for the notion of symmetry for a function of one variable, and thus we can use the following definition.

Definition 2. If the equality $v(-x) = \pm v(x)$ holds, then the function $v(x)$ is called symmetric. It should be noted that if in Eq. (1), the integral is as follows

$$\varphi(x, t) = \int_{t}^{x} k(x, t, s, y(s))ds, \tag{3}$$

then the notion of symmetry for the function $\varphi(x, t)$ can be determined by the classical definition of a symmetric function of two variables.

It should be noted that the symmetry of the function $b(x, s)$ does not imply the symmetry of the function $a(x, s, y(s))$. Indeed, if we suppose that $b(x, s) = \exp(x + s)$ and $y(s) = \exp(-s)$, then from the equality $a(x, s, y(s)) = \exp(s)$, it follows that the function $a(x, s, y(s))$ is not symmetrical.

Given the above, it is logical to find the notion of symmetry for the integral in the following form.

Definition 3. The Volterra integral equation (1), or the following Fredholm integral equation

$$y(x) = f(x) + \int_{a}^{b} k(x, s, y(s))ds, \quad x \in [a, b]$$

is symmetric if the kernel of the integral $k(x, s, y)$ is symmetric in the area as a function $\psi(x, s) = k(x, s, y(s))$ of two variables. However, the use of this method to study the integral equation (1) may sometimes lead to certain difficulties, which are related to the solution of the integral equation. In these cases, we can use Definition 1. When studying symmetric integral equations, it is interesting to compare the results obtained based on the symmetry and asymmetry of numerical methods. Therefore, in the next section, we consider the construction of symmetric numerical methods and describe a class of asymmetric methods.

2. Some approaches for constructing symmetrical multi-step methods

In general, integral equations of the Volterra type are solved using methods based on the successful application of ordinary differential equations (ODEs). Many classes of these methods can be employed to solve ODEs (e.g., [17–26]), including symmetrical and asymmetrical methods.

First, it is necessary to define the concept of symmetry for numerical methods and integral equations. Thus, we propose to use Definition 2 as a concept to consider the symmetry of real functions for one variable.

Processes for solving various equations often use approximation methods. Thus, the symmetry of the method employed to investigate the problem needs to apply a concept of symmetry. We note that there is no exact mathematical definition for the concept of the symmetry of integral equations. However, there have been some attempts to define the symmetry of integral equations. For example, the notion of symmetry for linear Fredholm integral equation was defined based on the symmetry of the kernel for integral equations [17–19]. Several definitions of the concept of symmetry have also been proposed for Runge–Kutta, Adams, and others methods [17, str. 232,354], [18, p.106]. Dahlquist [19] proposed a definition of symmetry for the following difference method

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i y'_{n+i}, \tag{4}$$

where (4) is symmetric if the following holds:

$$\alpha_j = -\alpha_{k-j}, \qquad \beta_j = \beta_{k-j} \quad (j = 0, 1, 2, \ldots, k).$$