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Inverse space-dependent force problems for the wave equation

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ABSTRACT

The determination of the displacement and the space-dependent force acting on a vibrating structure from measured final or time-average displacement observation is thoroughly investigated. Several aspects related to the existence and uniqueness of a solution of the linear but ill-posed inverse force problems are highlighted. After that, in order to capture the solution a variational formulation is proposed and the gradient of the least-squares functional that is minimized is rigorously and explicitly derived. Numerical results obtained using the Landweber method and the conjugate gradient method are presented and discussed illustrating the convergence of the iterative procedures for exact input data. Furthermore, for noisy data the semi-convergence phenomenon appears, as expected, and stability is restored by stopping the iterations according to the discrepancy principle criterion once the residual becomes close to the amount of noise. The present investigation will be significant to researchers concerned with wave propagation and control of vibrating structures.

1. Introduction

We consider the problem of force identification from measured data for the hyperbolic wave equation. This inverse formulation is significant to modelling several practical applications related to unknown force loads and control. Because part of the cause of the physical phenomenon is unknown one has to compensate for this lack of information by measuring an appropriate part of the effect. What quantity to measure is the delicate choice/constraint when formulating inverse problems, but a proper formulation would be able to ensure that the unknown force can be uniquely retrieved from the proposed additional measurements. However, stability can in general not be ensured.

Prior to this study, the reconstruction of a space-dependent force in the wave equation from Cauchy data measurements of both displacement and its normal derivative on the boundary has been attempted in several studies, e.g. [1–3]. Although the uniqueness of a solution still holds, [4–6], this inverse formulation is, as expected, improperly posed because the unknown output force $f(x)$ depends on x in the domain Ω, whilst the known input data, say u and $\partial_n u$, depend on (x, t) on the boundary $\partial\Omega \times (0, T)$. Therefore, it seems more natural to measure instead information about the displacement $u(x, t)$ for $x \in \Omega$ and time $t = T$, or the time-averaged displacement $\int_0^T u(x, t)dt$ for $x \in \Omega$. This way, the output–input mapping satisfies the meta-theorem that the overposed data and the unknown force function lie in the same direction, [7]. This spacewise-dependent force $f(x)$ identification from the upper-base spacewise dependent displacement measurement $u(x, T)$ has been investigated theoretically in Section 8.2 of [8], where the uniqueness of a solution was proved. For other

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wave related force identification studies which use the final time displacement data we refer to [9] which employs a weak solution approach for a relatively general inverse problem with a highly non-unique solution, and to [10] which nicely introduces a quasi-nonlinearity in the governing wave equation to resolve the non-uniqueness of a solution. The other inverse problem generated by the measurement of the time-averaged displacement $\int_0^T u(x,t)dt$ which we investigate in our study is new. Essentially, the same inverse problem with unknown spacewise dependent right-hand side source in the governing equation arises also for the parabolic heat equation in the thermal field, see [11,12].

The plan of the paper is as follows. Section 2 introduces the inverse problem formulations, whilst Sections 3 and 4 highlight several issues related to the existence, uniqueness and stability of a solution to the direct and inverse problems, respectively. Section 5 presents the variational formulations of the inverse problems under investigation and derives explicitly the expressions for the gradients of the least-squares functionals which are minimized. Section 6 describes the iterative Landweber method accommodated and applied in order to obtain regularized stable solutions, whilst Section 7 illustrates and discusses extensive numerical results in one dimension, for the recovery of smooth as well as non-smooth force functions. Furthermore, the conjugate gradient method (CGM) is also described and employed in one of the examples. A numerical extension to two-dimensions is presented in Section 8 and finally, conclusions are presented in Section 9.

2. Problem formulation

Assume that we have a medium, denoted by Ω, occupying a bounded sufficiently smooth domain in $\mathbb{R}^n$, where $n \geq 1$. The boundary of Ω is denoted by $\partial\Omega$, and we define the space–time cylinder $Q_T = \Omega \times (0, T)$, where $T > 0$. We wish to find the displacement $u(x, t)$ and the force $f(x)$ in the hyperbolic wave equation

$$u_{tt} - \mathcal{L}u = f(x)g(x,t) + \chi(x,t) =: F(x,t) \quad \text{in } Q_T, \tag{1}$$

where g and χ are given functions, and, in general, for a homogeneous medium we have $\mathcal{L} = \Delta$ the Laplacian operator. For inhomogeneous media, we can have $\mathcal{L}u = c(x)\Delta u$, or $\nabla \cdot (K(x)\nabla u)$, where c and K are given positive material properties, [13].

Eq. (1) has to be solved subject to prescribed initial conditions

$$u(x,0) = \varphi(x) \quad x \in \Omega, \tag{2}$$

$$u_t(x,0) = \psi(x) \quad x \in \Omega, \tag{3}$$

prescribed homogeneous Dirichlet boundary conditions,

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{4}$$

and the additional final displacement measurement

$$u(x,T) = u_T(x), \quad x \in \Omega, \tag{5}$$

or, the time-average displacement measurement

$$\int_0^T \omega(t)u(x,t)dt = U_T(x), \quad x \in \Omega, \tag{6}$$

where ω is a given weight function.

3. Direct problem

Well-posedness of (1)–(4), when F is given can be shown in various spaces. Natural in our case is to work with (weak) solutions in a form that makes Green's formula (integration by parts) valid. Thus, we assume that $F \in L^2(0, T; L^2(\Omega))$ (this space is equivalent to $L^2(Q_T)$), $\varphi \in H_0^1(\Omega)$ and $\psi \in L^2(\Omega)$. Then there exists a unique solution satisfying $u \in C([0, T]; H_0^1(\Omega))$, $u_t \in C([0, T]; L^2(\Omega))$ with (1) satisfied in the distributional sense, (2)–(3) satisfied almost everywhere and (4) satisfied since $u(\cdot, t)$ belongs to $H_0^1(\Omega)$. In fact, one has $u_{tt} \in L^2(0, T; H^{-1}(\Omega))$. Moreover, we have the estimate

$$\max_{0\leq t\leq T}\left(\|u(.,t)\|_{H_0^1(\Omega)} + \|u_t(.,t)\|_{L^2(\Omega)}\right) + \|u_{tt}\|_{L^2(0,T;H^{-1}(\Omega))} \leq \mathrm{C}\left(\|F\|_{L^2(0,T;L^2(\Omega))} + \|\varphi\|_{H_0^1(\Omega)} + \|\psi\|_{L^2(\Omega)}\right), \tag{7}$$

for some positive constant C. For the well-posedness and estimate, see Chapter 3.8 of [14] and Chapter 5.1 of [15], where also regularity of the solution is investigated.

Similar considerations can be done for the formally adjoint backward hyperbolic problem to (1)–(4),

$$v_{tt} - \mathcal{L}^*v = G(x,t), \quad (x,t) \in Q_T, \tag{8}$$

$$v(x,T) = \zeta(x), \qquad v_t(x,T) = \xi(x), \quad x \in \Omega, \tag{9}$$

$$v(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{10}$$

where $\mathcal{L}^*$ is the adjoint of $\mathcal{L}$.