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Numerical problems with the Pascal triangle in moment computation

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ABSTRACT

Moments are important characteristics of digital signals and images and are commonly used for their description and classification. When calculating the moments and their derived functions numerically, we face, among other numerical problems studied in the literature, certain instabilities which are connected with the properties of Pascal triangle. The Pascal triangle appears in moment computation in various forms whenever we have to deal with binomial powers. In this paper, we investigate the reasons for these instabilities in three particular cases—central moments, complex moments, and moment blur invariants. While in the first two cases this phenomenon is tolerable, in the third one it causes serious numerical problems. We analyze these problems and show that they can be partially overcome by choosing an appropriate polynomial basis.

1. Introduction

Moments are scalar quantities which have been used for more than hundred years to characterize a (possibly multidimensional) function and to capture its significant features. They have been widely used in statistics for the description of the shape of a probability density function and in classic rigid-body mechanics to measure the mass distribution of a body. From the mathematical point of view, moments are projections of a function onto a polynomial basis (in the same way that the Fourier transform is a projection onto a basis of harmonic functions).

Moments were first introduced to the pattern recognition and image processing community in 1962, when Hu [1] employed the results of the classical theory of algebraic invariants [2–5] and derived the first moment-based characteristics (features) suitable for object description and recognition. Since that time, this field has undergone significant development. The study of moments has formed a well-established area of image recognition with thousands of relevant papers and several survey monographs [6–10] and has become one of the most frequently used features in image analysis.

A general definition of a moment in d dimensions is as follows. Let $\{b_{\mathbf{k}}(\mathbf{x})\}$ be a d-variable polynomial basis of the space of image functions defined on $D \subset \mathbb{R}^d$ and let $\mathbf{k} = (k_1, \ldots, k_d)$ be a multi-index of non-negative integers which show the highest power of the respective variables in $b_{\mathbf{k}}(\mathbf{x})$. Then the *general moment* $M_{\mathbf{k}}^{(f)}$ of image f is defined as

$$M_{\mathbf{k}}^{(f)} = \int_D b_{\mathbf{k}}(\mathbf{x}) f(\mathbf{x}) \mathrm{d}\mathbf{x}. \tag{1}$$

The number $|\mathbf{k}| = \sum_{j=1}^{d} k_j$ is called the *order* of the moment. We omit the superscript (f) when there is no ambiguity.

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The properties of the moments depend on the choice of the basis $\{b_{\mathbf{k}}(\mathbf{x})\}$. The most common choice is a standard power basis

$$b_{\mathbf{k}}(\mathbf{x}) = \mathbf{x}^{\mathbf{k}} = x_1^{k_1} \dots x_d^{k_d}$$

which leads to *geometric moments*

$$m_{\mathbf{k}} = \int_D \mathbf{x}^{\mathbf{k}} f(\mathbf{x}) \mathrm{d}\mathbf{x}. \tag{2}$$

In addition to that, numerous orthogonal bases have been used in the literature (see [8] for a survey).

The moments themselves are rarely used directly for object recognition because they depend on the position, orientation, size, and many other variations of the object. To overcome this, one can create certain functions of moments which stay constant under certain group of transformations. These functions are called *moment invariants*. A number of various moment invariants have been reported in the literature—rotation invariants, similarity invariants, affine invariants, and blur invariants are the most important ones (see [8] for a survey and detailed forms of individual invariants).

Although the theory of all moment invariants has been developed in a continuous domain because of the comfortable mathematical tools available, our practical interest is in the domain of discrete (digital) signals and images. The transition from the continuous to the discrete domain entails an approximation of the integral in (1) by a sum. In the simplest case, a zero-order approximation is used and (1) turns to the form (we show a 1D case for simplicity)

$$M_k = \sum_{j=1}^{N} b_k(x_j) f(x_j) \tag{3}$$

where the evaluation points are commonly chosen $x_j = j$. Regardless of what approximation scheme has been applied, this transition always induces errors of several kinds. Some of them are firmly connected with the sampling and quantization errors of the image and have been thoroughly analyzed for instance in [7].

Another group of errors is connected with the numerical evaluation of a sum such as that in (3) and originates from the finite precision of the computer arithmetic. Although they might be by several orders higher than the sampling errors (especially if an unstable numerical algorithm has been applied), they have not been systematically studied and fully explored yet. Most papers on numerical moment calculations have been focused on fast algorithms rather than on error analysis [11,12]. If the authors had observed some instability in the experiments, they mostly explained it as a moment sensitivity to additive noise in the image or as a consequence of using very large values when working with the powers. They tried, with partial success, to overcome these problems by changing the polynomial basis to Legendre [13], Zernike [13,14], Pseudo-Zernike [15], Gauss–Hermite [16], Chebyshev [17], Krawtchouk [18], and other special bases [19,20].

In this paper we investigate a specific source of errors: the poor condition of a Pascal matrix (11). We show that the Pascal matrix (and forms derived from it) often appears when evaluating polynomial moments and moment invariants. We demonstrate that in some cases it introduces serious numerical errors while in other, seemingly very similar, cases its impact is much less significant. This aspect of moment calculations has never been investigated and errors of this kind have, in the past, been misinterpreted.

The paper is organized as follows. In Section 2 we recall three important classes of moments, and their invariants, where Pascal-like matrices appear in their evaluation. Section 3 presents a general overview of numerical algebra on the given problem. In Section 4 we briefly discuss the importance of choosing the suitable domain for the calculation of geometric moments (that is, choosing x_j in (3)), then we study how the Pascal triangle influences the transformations between geometric, central and complex moments. We describe the effect of combining the Pascal triangle with a Toeplitz matrix carrying the centroid information and, finally, we show that to calculate complex moments from the geometric moments, a 2D version of Pascal triangle leads to better numerical accuracy than might be expected. The stability of all these calculations can be studied by elementary means, that is through the numerical condition of the linear transforms involved. On the other hand, the evaluation of *blur invariants* leads to a numerical misbehavior of such magnitude that cannot be explained just by the numerical condition of the underlying linear system. In Section 5 we demonstrate, analyze and explain this instability, caused inherently by the Pascal triangle. We demonstrate that the error of this kind is actually very serious because it renders higher order moments useless for the purpose of discriminating between blurred images. We show how it can be partially overcome. This is the main contribution of the paper. Section 6 contains a summary and discussion.

2. Central moments, complex moments and blur invariants

In this section, we introduce three examples of derived moments, where various versions of the Pascal triangle appear. The influence of the Pascal triangle on the numerical calculations will be investigated later in the paper. We limit ourselves to the 1D and 2D cases for the sake of simplicity. The extension into 3D or even to higher dimension (in case of blur invariants) is, in principle, straightforward.

The most common moments are those with respect to the basis composed of the power monomials $\{x^p\}$ in 1D and $\{x^p y^q\}$ in 2D. They are called *geometric moments* and in the 2D continuous domain are defined as

$$m_{pq} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^p y^q f(x, y) dx dy. \tag{4}$$