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Second-order asymptotic algorithm for heat conduction problems of periodic composite materials in curvilinear coordinates

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ABSTRACT

A new second-order two-scale (SOTS) asymptotic analysis method is presented for the heat conduction problems concerning composite materials with periodic configuration under the coordinate transformation. The heat conduction problems are solved on the transformed regular domain with quasi-periodic structure in the general curvilinear coordinate system. By the asymptotic expansion, the cell problems, effective material coefficients and homogenized heat conduction problems are obtained successively. The main characteristic of the approximate model is that each cell problem defined on the microscopic cell domain is associated with the macroscopic coordinate. The error estimation of the asymptotic analysis method is established on some regularity hypothesis. Some common coordinate transformations are discussed and the reduced SOTS solutions are presented. Especially by considering the general one-dimensional problem, the explicit expressions of the SOTS solutions are derived and stronger error estimation is presented. Finally, the corresponding finite element algorithms are presented and numerical results are analyzed. The numerical errors presented agree well with the theoretical prediction, which demonstrate the effectiveness of the second-order asymptotic analysis method. By the coordinate transformation, the asymptotic analysis method can be extended to more general domain with periodic microscopic structures.

1. Introduction

Composite materials are made from two or more constituent materials with significantly different physical or chemical properties, that when combined, produce a material with characteristics different from the individual components. They can overcome the defects of a single material and expand the application scope. With the rapid development of aeronautic and aerospace engineering, composites are widely used for manufacturing aircraft wing, satellite antenna and its supporting structure, solar wing and shell, and the Thermal Protection System (TPS) for the space vehicle, etc., because of the good thermal stability, high specific stiffness and strength. The composites are mostly heterogeneous with complex micro structures. Developing an effective method to simulate and predict the thermal and mechanical performance is of importance and practical significance and is necessary for the design and manufacture of the composites.

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The evaluation of the heat transferring for the composite materials mainly involves studying the properties of the second-order elliptic equations with rapidly oscillating coefficients and many related basic scientific problems, including multi-scale correlation model, and high-performance multi-scale algorithms should be considered. In the recent decades, the researches on the multi-scale algorithm have attracted the attention of many authors. Mathematically, Bensoussan et al. [1] introduced the asymptotic expansion, and applied this idea to various physical and mechanical problems, and gave the theoretical results of the converging behavior. Oleinik et al. [2] studied the homogenization problem and proved the well-known $\varepsilon^{1/2}$ error estimation. The homogenization method is investigated and extended to the steady or transient heat conduction problem, wave propagation [3], optimal control [4], and different differential operators and integral functions [5], and perforated structures [6]. The vibration theory of the heterogeneous media is considered by Sanchez-Palencia [7]. Allaire [8] considered the homogenization problem by introducing the well-known two-scale convergence method, and recently developed the asymptotic expansion of a coupled conduction–radiation heat transfer problem [9]. Based on this research, various multi-scale methods have been proposed [10,11], but only the first-order asymptotic expansions are considered. Recently, Cui and Cao et al. [12–17] introduced the Second-Order Two-Scale (SOTS) analysis method to predict physical and mechanical behaviors of composite and perforated materials more accurately by considering the second-order correctors. Especially by choosing proper representative cell domain, higher-order convergence and more accurate results were obtained [18]. Su and Ma [19–21] extended the SOTS method to heat conduction, linear elasticity and thermo-elastic problems for quasi-periodic structures. Wang [22] developed the SOTS method for bending behavior analysis of composite plate. Another remarkable work for the heat conduction of the periodic or porous composite material is the SOTS asymptotic expansion integrating conduction, convection and radiation is obtained by Yang, Zhang and Ma [23–26]. Incorporated with the thermo-elastic response of the composites, Feng and Wan [27,28] obtained the multi-scale second-order solutions for the quasi-static and dynamic thermo-elastic problems. Yang [29] used the statistical SOTS method to compute the dynamic thermo-mechanical coupled response of random particulate composites.

According to most literatures, although the asymptotic studies of the heat conduction problem are extended to the porous materials [18], random distributed structure [29], the coupled conduction–radiation condition [9,24,25], etc., the numerical simulations were often tested and carried out only in the regular domains, rectangle or cuboid for example, and did not extend to more general and irregular domains. As we all know, the structures made of composites are not always regular, and the cylinder, ball, or fan-shaped structures, etc. are often used in the engineering. The hyperbolic plates and shells are applied in the TPS for the space vehicle with complicated microscopic structures and the SOTS method cannot be applied to these kinds of structures directly due to their curved profiles. Based on the idea of coordinate transformation that is often adopted in the finite difference computation, we try to find a proper transformation from the regular domain to this particular domain so as to apply the SOTS method in the regular domain and obtain the asymptotic solutions. Then, the heat transfer problem in the original domain becomes quasi-periodic problem after the transformation. Finally, we obtain the real asymptotic solutions by the inverse transformation. The mesh generation method can also be used to make the direct mapping between the two domains. By using this idea, the SOTS method can be extended to more arbitrary domains.

The remaining part of this paper is outlined as follows: The heat conduction problem in the curvilinear coordinate system and the coordinate transformation are presented in Section 2. The SOTS analysis method is discussed in Section 3, including the cell problems, homogenized solutions and coefficients, the error estimation and some specific transformations. The finite element algorithm is proposed in Section 4. Three numerical examples are tested and discussed in Section 5, followed by conclusions and future work in Section 6. Throughout this paper, the common notations of Sobolev space and the convention of summation on repeated indices are used, and the letters in bold represent the matrix or vector functions in the formulation. By $O(\varepsilon^k),\ k \in \mathbb{N}$, we denote there exists a constant c independent of ε such that $|O(\varepsilon^k)| \leq c\varepsilon^k$.

2. Governing equation

Without loss of generality, the coordinate transformation is illustrated in Fig. 1 in the two-dimensional domain. The structure occupying the domain Ω in Fig. 1(a) is quasi-periodically distributed in the conventional Cartesian coordinate $\boldsymbol{x}$, which is to say that although we can choose a representative cell Υ shown in Fig. 1(c) to characterize the microscopic feature of the structure, the cells in the macroscopic domain may be bent or twisted such that all of them are different from each other. For this structure, the asymptotic expansion method [1–4] cannot be adopted directly, as it is only effective when the cells in the composite materials are periodically arranged along the axis. However, by choosing proper coordinate transformation,

$$\boldsymbol{\xi} = \boldsymbol{L}(\boldsymbol{x}), \tag{1}$$

this quasi-periodic domain Ω is transformed into periodic domain Θ shown in Fig. 1(b) in the general curvilinear coordinate $\boldsymbol{\xi}$. Then the asymptotic expansion method can be applied to analyze the heat conduction performance of the structure, and after obtaining the asymptotic expansion of the temperature, by the inverse transformation

$$\boldsymbol{x} = \boldsymbol{L}^{-1}(\boldsymbol{\xi}), \tag{2}$$

the asymptotic behavior of the temperature in the original domain Ω can be simulated and predicted. To make this transformation well defined, we assume the transformation (1) is proper, i.e. the function $\boldsymbol{L}(\boldsymbol{x})$ is monotonous bounded, continuous and its first partial derivative is also continuous, in which case the inverse transformation always exists.