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A novel method for solving second order fractional eigenvalue problems

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ABSTRACT

The paper presents a new numerical method for solving eigenvalue problems for fractional differential equations. It combines two techniques: the method of external excitation (MEE) and the backward substitution method (BSM). The first one is a mathematical model of physical measurements when a mechanical, electrical or acoustic system is excited by some source and resonant frequencies can be determined by using the growth of the amplitude of oscillations near these frequencies. The BSM consists of replacing the original equation by an approximate equation which has an exact analytic solution with a set of free parameters. These free parameters are determined by the use of the collocation procedure. Some examples are given to demonstrate the validity and applicability of the new method and a comparison is made with the existing results. The numerical results show that the proposed method is of a high accuracy and is efficient for solving of a wide class of eigenvalue problems.

1. Introduction

In this paper we present a novel method for solving fractional eigenvalue problems of the second order:

$$L[u] + \lambda p(x)\, u(x) \equiv \alpha(x)\, D^{(\nu)} u(x) + \sum_{k=1}^{K} \beta_k(x)\, D^{(\nu_k)} u(x) + \lambda p(x)\, u(x) = 0, \quad 0 \le x \le 1, \tag{1}$$

$$a_0 u(0) + a_1 u^{(1)}(0) = 0, \qquad b_0 u(1) + b_1 u^{(1)}(1) = 0, \tag{2}$$

where $1 < \nu \le 2,\ 0 < \nu_k \le 1$, the coefficients $\alpha(x), \beta_k(x), p(x)$ are known smooth enough functions and a_0, a_1, b_0, b_1 are appropriate constants.

Throughout the paper we consider the fractional Caputo derivatives which are defined as follows [1–3]:

$$D^{(\nu)} f(x) = \begin{cases} \dfrac{1}{\Gamma(n-\nu)} \displaystyle\int_0^x \frac{f^{(n)}(t)}{(x-t)^{\nu-n+1}}, & n-1 < \nu < n, \\ f^{(n)}(x), & \nu = n, \end{cases}, \tag{3}$$

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where $n \in \mathcal{N} = \{1, 2, \ldots\}$ is the set of positive integers and $\Gamma(z)$ denotes the gamma function. In particular for power functions we get:

$$D^{(\nu)}x^p = \begin{cases} 0, & \text{if } p \in \mathcal{N}_0 \text{ and } p < n, \\ \dfrac{\Gamma(p+1)}{\Gamma(p+1-\nu)}x^{p-\nu}, & \text{if } p \in \mathcal{N}_0 \text{ and } p \geq n \text{ or } p \notin \mathcal{N}_0 \text{ and } p > n-1, \end{cases} \tag{4}$$

where $\mathcal{N}_0 = \{0, 1, 2, \ldots\}$ is the set of nonnegative integers. This formula is widely used throughout the paper.

Fractional differential equations and fractional integro-differential equations arise in various areas of science and engineering. The advantages of FDEs become apparent in modeling electrical properties of real materials, as well as in description of blood flow, and in many other fields (see, for example, [4,5] and references cited therein). Due to the growing applications considerable attention has been given to the numerical solutions of FDEs. The review of the numerical techniques developed in the early 2000s can be found in [1].

The shifted Chebyshev and Jacobi polynomials are used for approximation of the solution in the methods developed in [6–8]. The polynomial approximation is used there together with tau and collocation spectral methods to find an approximate solution for multi-term linear and nonlinear FDEs. A similar technique based on the use of the shifted Legendre polynomials was introduced by Bhrawy et al. in [9] for solving multi-order fractional differential equations with constant coefficients. The collocation-shooting method for solving fractional boundary value problems of the second order was proposed in [10]. In [11] the method based on the Legendre wavelet operational matrix of fractional order integrations was applied for solving FDEs by Rehman and Khan. The direct solution technique for multi-order FDEs with variable coefficients using the quadrature shifted Legendre tau method was developed in [12]. Maleki et al. proposed an adaptive pseudospectral method for solving a class of multiterm fractional boundary value problems in [13]. Pedas and Tamme developed the spline collocation methods for solving FDEs in [14–16]. A modification of this technique was proposed by Kolk et al. in [17]. A nonpolynomial collocation method for solving a class of initial and boundary value problems was proposed in [18,19]. This technique is based on the transformation of the original problem into the integral equation of the Volterra and Fredholm type. A spectral collocation method applying the Bessel functions was proposed by Parand and Nikarya in [20] for solving FDEs and FIDs. In [21] Rehman and Khan proposed a numerical scheme, based on the Haar wavelet operational matrices for solving linear multi-point BVPs for FDEs. Recently, the reproducing kernel method was proposed by Li and Wu for solving FDE with nonlocal boundary conditions [22]. In [23] Rawashdeh studied the numerical solution of FIDEs by polynomial spline functions. The collocation method based on the use of the operational matrix of derivative for general Jacobi's orthogonal polynomials was proposed by Eslahchi et al. in [24] for solving nonlinear FIDEs. A similar approach based on the use of the Legendre wavelets operational matrix method was proposed by Meng et al. in [25]. The method which is based on the truncated Taylor expansions for solving linear FIDEs of the Fredholm type was proposed in [26].

At the same time only a few number of papers are devoted to fractional eigenvalue problems. The method of solution based on utilizing the series solution was proposed by Hajji et al. in [27]. The FDE is converted into a linear system of algebraic equations and then the eigenvalues are calculated as roots of the characteristic polynomial. The Adomian decomposition method and the homotopy analysis method were proposed by Al-Mdallal for this goal in [28,29] correspondingly. The homotopy analysis method for fractional Sturm–Liouville eigenvalue problems was also proposed by Abbasbandy and Shirzadi in [30]. The eigenvalue problems for the FDEs $D^{(\nu)}u(x) + \lambda u(x) = 0,\ 0 < \nu \leq 2$ with different classes of boundary conditions were investigated in detail in [31] by Duan et al. by using the general solutions and the theory of the Mittag-Leffler functions. Recently the augmented-RBF method has been proposed to solve fractional eigenvalue problems by Antunes and Ferreira in [32].

For solving the eigenvalue problem (1), (2) we combine two techniques: the method of external excitation (MEE) [33,34] and the backward substitution method (BSM) [35,36].

Let us consider the eigenvalue problem

$$L[u] + \lambda p(x)u = 0, \qquad B_0[u(0)] = 0, \qquad B_1[u(1)] = 0, \tag{5}$$

where $L[\ldots]$ is a linear differential operator and $B_0[\ldots]$, $B_1[\ldots]$ are operators of the boundary conditions. Let $u_e(x)$ be a smooth enough function defined in the interval $[0, 1]$ named below as the *exciting field*. If the *response field* $u_r(x)$ is a solution of the following boundary value problem (BVP)

$$L[u_r] + \lambda p(x)u_r = -L[u_e] - \lambda p(x)u_e, \tag{6}$$

$$B_0[u_r(0)] = -B_0[u_e(0)], \qquad B_1[u_r(1)] = -B_1[u_e(1)], \tag{7}$$

then the sum $u(x, \lambda) = u_r + u_e$ satisfies the initial problem (5). Let $F(\lambda)$ be some norm of the solution $u(x)$. This function of λ has maxima at the eigenvalues and, under some conditions described below, can be used for their determining. The MEE technique is convenient for determining some first eigenvalues of the system which are often of the most interest for engineering applications. Applying it, we transform the original fractional eigenvalue problem (1), (2) into a sequence of fractional boundary value problems (FBVPs) (6), (7), which depend on the spectral parameter λ. So, an effective method for solving the FBVPs should be offered. The method presented in the paper for this goal is a development of the numerical technique proposed earlier in [35,36]. The general scheme is as follows. Let us write the governing FDE in the form:

$$L[u] = F(x, u), \tag{8}$$