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# The method of fundamental solutions applied to boundary eigenvalue problems

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ABSTRACT

We develop methods based on fundamental solutions to compute the Steklov, Wentzell and Laplace–Beltrami eigenvalues in the context of shape optimization. In the class of smooth simply connected two dimensional domains the numerical method is accurate and fast. A theoretical error bound is given along with comparisons with mesh-based methods. We illustrate the use of this method in the study of a wide class of shape optimization problems in two dimensions. We extend the method to the computation of the Laplace–Beltrami eigenvalues on surfaces and we investigate some spectral optimal partitioning problems.



## 1. Introduction

The purpose of this article is to provide some tools which facilitate the numerical study of some shape optimization problems. Such problems consist of minimizing or maximizing a certain quantity which depends on the domain geometry. The cost function which is to be optimized may depend on the geometric properties of the domain (perimeter, area) or on some more complex quantities given, for example, by some partial differential equations (eigenvalues, integral energies). Finding explicitly solutions to shape optimization problems may be difficult or even impossible in some cases. Thus, having efficient numerical methods which can allow the study of shape optimization problems is an important issue.

This article treats the numerical optimization of functionals which depend on eigenvalue problems defined on the boundary of the considered domain. The numerical algorithms presented in the sequel allow the computation of the eigenvalues of the Steklov, Wentzell and Laplace–Beltrami spectra. In order to compute these eigenvalues for a given domain we develop a method based on fundamental solutions. This type of methods has been introduced in [1] and has been used by Antunes and Alves in the study of various eigenvalue problems [2–4]. The advantage of such a method is the fact that there is no need for a mesh generation at each function evaluation and for a large class of domains the corresponding eigenvalue computation is fast. This fact allows an important time economy if we wish to use the algorithm for numerical shape optimization. Another advantage is that the method based on fundamental solutions is precise. We provide a theoretical result which estimates this error in the case of the Steklov and Wentzell eigenvalues.

There are a few works which present applications of such mesh-less computation methods to the numerical study of shape optimization problems. Among these we mention the minimization of the Laplace Dirichlet eigenvalues by Antunes and Freitas in [5], where the authors also use the method of fundamental solutions. Other functionals depending on the

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Dirichlet–Laplace eigenvalues have been studied by Antunes in [6], using fundamental solutions, and by Osting in [7] using the method of particular solutions for the eigenvalue computation. Optimal convex combinations of Dirichlet–Laplace eigenvalues have been studied in [8,9] using the MpsPack Matlab toolbox [10] for the eigenvalue computation. In a recent result of Akhmetgaliyev, Kao and Osting [11] the authors provide a numerical method based on a single layer potential in order to compute the Steklov eigenvalues on a two dimensional domain. They also optimize numerically the Steklov eigenvalues under area constraint in dimension two.

We recall that for $\Omega \subset \mathbb{R}^n$ an open set with Lipschitz boundary we can define the **Steklov eigenvalues** as the real values $\sigma$ for which the following problem has a non trivial solution:

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = \sigma u & \text{on } \partial\Omega. \end{cases}$$

It is easy to see that 0 is a Steklov eigenvalue corresponding to constant functions. This problem has a discrete spectrum given by an increasing sequence

$$0 = \sigma_0(\Omega) \le \sigma_1(\Omega) \le \sigma_2(\Omega) \le \sigma_3(\Omega) \le \cdots \to +\infty.$$

As usual, we can provide a variational characterization using Rayleigh quotients

$$\sigma_n(\Omega) = \inf_{S_n} \sup_{u \in S_n \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\int_{\partial\Omega} u^2 d\sigma}, \quad n = 1, 2, \ldots$$

where $S_n$ is an $n$ dimensional linear subspace of $H^1(\Omega) \cap \{\int_{\partial\Omega} u = 0\}$. This variational formulation allows us to deduce immediately the behavior of the Steklov eigenvalues under homotheties: $\sigma_k(t\Omega) = \sigma_k(\Omega)/t$ for all $t > 0$.

The study of the Steklov spectrum is and has been a very active field of research. We cite here some notable results concerning the optimization of these eigenvalues. Weinstock proved in [12] that the first non-zero Steklov eigenvalue is maximized by the disk in the class of simply connected two dimensional domains with fixed perimeter. This result was generalized in further directions by Hersch, Payne and Schiffer in [13]. Brock proved in [14] that the first non-zero Steklov eigenvalue is maximized by the ball in every dimension when considering a volume constraint, without any restrictions on the topology of the domain. Other results concerning the optimization of the Steklov spectrum under perimeter constraint are presented in [15]. As underlined in [16], all these optimization results are proved by precisely identifying the optimal shape and then proving that the shape is indeed the desired optimizer. There are cases where the optimal shape cannot be determined explicitly and thus we have a good motivation to develop numerical tools. Concerning the existence of the optimal shapes, some general results are given in [16] in the class of convex sets or the class of sets which satisfy a uniform $\varepsilon$-cone property. General results concerning the existence of optimal shapes for the Steklov problem in the class of simply connected domains can be found in [17].

The **Wentzell spectrum** consists of the real values $\lambda$ for which the following problem has non-trivial solutions:

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ -\beta \Delta_\tau u + \partial u_n = \lambda u & \text{on } \partial\Omega. \end{cases}$$

We note that the Steklov case corresponds to $\beta = 0$. Although it is possible to study this problem for every real number $\beta$, we restrict ourselves to the case $\beta \ge 0$. The Wentzell spectrum is discrete and is given by an increasing sequence denoted

$$0 = \lambda_{0,\beta}(\Omega) \le \lambda_{1,\beta}(\Omega) \le \lambda_{2,\beta}(\Omega) \le \lambda_{3,\beta}(\Omega) \le \cdots \to +\infty.$$

The case of the Wentzell problem has been recently studied in [18] where the authors prove that the ball maximizes locally the first non-zero Wentzell eigenvalue under volume constraint in the class of sets homeomorphic to a ball. It is conjectured that the ball is the global minimizer in the same class of admissible sets. We are able to numerically validate this result in the two dimensional case. We also illustrate the fact that without the topological assumption the conjecture is false.

The second class of problems we study in this article concerns the partitions of a three dimensional surface which minimize the sum of the first Laplace–Beltrami eigenvalues with Dirichlet boundary conditions. It is well known that the **spectrum of Laplace–Beltrami operator** with Dirichlet boundary conditions of a subset $\omega \subset \partial\Omega$ consists of the values $\lambda$ such that the problem

$$\begin{cases} -\Delta_\tau u = \lambda u & \text{in } \omega \\ u = 0 & \text{on } \partial\omega \end{cases}$$

has non trivial solutions. This spectrum is discrete and consists of an increasing sequence denoted

$$0 = \lambda_0^{LB}(\omega) \le \lambda_1^{LB}(\omega) \le \lambda_1^{LB}(\omega) \le \cdots \to +\infty.$$

It is known that if $\omega$ is homeomorphic to the three dimensional euclidean sphere, then $\lambda_1^{LB}(\omega)$ is maximized by the sphere with the same surface area. For the two dimensional case we know that $\lambda_1^{LB}(\partial\Omega) \le \lambda_1^{LB}(\partial B)$ where $B$ is a disk of same area as $\Omega$. For a proof of this fact we refer to [18, Section 2.1]. The Wentzell and Laplace–Beltrami eigenvalues are related by the property $\lim_{\beta\to\infty} \lambda_{k,\beta}(\Omega)/\beta = \lambda_k^{LB}(\partial\Omega)$. For a proof we refer to [18, Section 2.1].