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Non standard finite difference scheme preserving dynamical properties

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ABSTRACT

We study the construction of a non-standard finite differences numerical scheme for a general class of two dimensional differential equations including several models in population dynamics using the idea of non-local approximation introduced by R. Mickens. We prove the convergence of the scheme, the unconditional, with respect to the discretization parameter, preservation of the fixed points of the continuous system and the preservation of their stability nature. Several numerical examples are given and comparison with usual numerical scheme (Euler, Runge–Kutta of order 2 or 4) is detailed.

1. Introduction

Differential equations are in general difficult to solve and study. In particular, for most of them we do not know explicit solutions. As a consequence, one is led to perform numerical experiments using some "integrators" such as the Euler or Runge–Kutta numerical scheme. The construction of these methods is based on approximation theory and focus on the way to produce finite representation of functions. Although crucial to obtain good agreements between a given solution and its approximation, it is far from being sufficient. Indeed, these numerical methods produce artefacts, i.e. numerical behaviour which is not present in the given model. Examples of these artefacts are: creation of ghost equilibrium points, change in the stability nature of existing equilibrium point or destruction of domain invariance, etc.

These issues are of course of fundamental importance and there is a way to solve it. Indeed, the artefacts produced by classical numerical methods are related to the non persistence of some important features of the dynamics generated by the differential equation. In particular, the qualitative theory of differential equations is mainly concerned with invariant objects such as equilibrium points and their dynamical properties such as stability or instability as well as other global properties such as domain invariance and variational structures. Therefore, an idea emerged to construct numerical schemes that do not focus on the approximation problems but deal with some dynamical information leading to what can be called *qualitative dynamical numerical scheme*.

This program was in fact mainly developed by R. Mickens in a series of papers (see [1–3]). In order to distinguish the new numerical scheme from the classical one, he coined the term nonstandard schemes for them.

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The aim of this paper is to introduce a nonstandard scheme concerning a class of differential equations that include all prey–predator models. The study of nonstandard scheme for prey–predator models is extensive but has been done only with specific form of the differential equations (see [4–6]). Our results generalize the one obtained by D.T. Dimitrov and H.V. Kojouharov in [4] and related works discussed in [7–9].

Precisely, we give a complete answer to the following questions and problems:

- Is the non-standard scheme convergent?
- For which value of the time step increment, do we recover the stability of the fixed points?
- To give a comparison between the numerical results obtained using the non-standard scheme and other well known methods (such as Euler, Runge–Kutta 2 or Runge–Kutta 4).

To our knowledge, these questions and problems are not solved in the previous cited paper. In this article, we prove convergence of the non-standard scheme which is constructed. A comparison between our scheme and Euler, Runge–Kutta 2 and 4 is also given. Precisely, classical problems related to the behaviour of these schemes with respect to equilibrium points, stability and positivity are discussed with respect to the time step increment. Such a comparison is not provided in the existing literature. Moreover, we prove that two of the third class of equilibrium points are preserved unconditionally with respect to the time-step increment. In comparison, in [7–9], the authors obtain only stability for a sufficiently small time-step which in fact follows directly from standard arguments in dynamical systems theory (see Section 6.4).

The plan of the paper is as follows:

In Section 2, we recall classical definitions of equilibrium points and their stability for discrete and continuous dynamical systems. Section 3 gives the definition of a non-standard finite difference scheme following R. Anguelov and J.M-S. Lubuma [10,11]. In Section 4, we introduce the class of two dimensional differential equations that we are considering and we study the positivity and the stability of the equilibrium points of this class of differential equations. In Section 5, we introduce the non-standard scheme associated to this system with results about the preservation of stability and positivity of the initial problem. In Section 6, we illustrate numerically the results on different models. Section 7 concludes the paper and provides some perspectives and comments.

2. Reminder about continuous/discrete dynamical systems

In this section, we remind classical results about continuous and discrete dynamical systems dealing with the *qualitative* behaviour of ordinary differential equations which will be studied both for our class of models and their discretization. We refer in particular to the book of S. Wiggins [12] for more details and proofs.

2.1. Vector fields

2.1.1. Equilibrium points and stability

Consider a general autonomous differential equation

$$\frac{dx(t)}{dt} = f(x(t)), \quad x \in \mathbb{R}^n, \tag{1}$$

where $f \in C^2(\mathbb{R}^m, \mathbb{R}^m)$ is called the *vector fields* associated to (1).

An *equilibrium solution* of (1) is a point $E \in \mathbb{R}^n$ such that $f(E) = 0$. We denote by $\mathcal{F}$ the set of equilibrium points of (1).

An important issue is to understand the dynamics of trajectories in the neighbourhood of a given equilibrium point. This is done through different notions of *stability*. In our model, we will use mainly the notion of *asymptotic stability* which is a stronger notion than the usual *Liapounov stability*.

Definition 2.1 (*Liapounov Stability*)**.** A solution $x(t)$ of (1) is said to be stable if, given $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that, for any other solution, $y(t)$, of (1) satisfying $\|x(t_0) - y(t_0)\| < \delta$, then $\|x(t) - y(t)\| < \epsilon$ for $t > t_0$, $t_0 \in \mathbb{R}$.

Our main concern will be asymptotic stability.

Definition 2.2 (*Asymptotic Stability*)**.** A solution $x(t)$ of (1) is said to be asymptotically stable if it is Liapounov stable and for any other solution, $y(t)$, of (1), there exists a constant $\delta > 0$ such that if $\|x(t_0) - y(t_0)\| < \delta$, then $\lim_{t\to+\infty} \|x(t) - y(t)\| = 0$.

For an equilibrium E, an important result is that asymptotic stability can be determined from the associated *linear system* defined by

$$\frac{dy}{dt} = Df(E)y, \tag{2}$$

where $Df(E)$ is the Jacobian of f evaluated at point E.

Precisely, we have (see [12], Theorem 1.2.5 p. 11):