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Exponential fitting Runge–Kutta methods for the delayed recruitment/renewal equation

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ABSTRACT

The so-called delayed recruitment/renewal equation provides the mathematical model in a diverse spectrum of practical applications and may become singularly perturbed when the time-lag is large relative to the reciprocal of the decay rate. In order to accurately capture its solution features numerically, we design a family of exponential fitting Runge–Kutta methods of collocation type to obtain the numerical approximation. The exponential fitting approximations are proved to have higher order of uniform accuracy. We demonstrate the efficiency of this family of exponential fitting Runge–Kutta methods for the delayed recruitment/renewal equation via application to some important problems.

1. Introduction

Delay differential equations (DDES) arise frequently in the mathematical modeling of a large variety of practical phenomena in the dynamical diseases, population dynamics, physics, economics, biosciences, engineering and control theory, in which the time evolution depends not only on present states but also on states at or near a given time in the past (see, e.g. [1–8]).

Many of the above applications involve the so-called delayed recruitment/renewal equation [9]

$$\frac{dx(\tau)}{d\tau} = -ax(\tau) + P(x(\tau - \tau_D)), \tag{1}$$

where x measures the amount or concentration of some substance while a is its decay rate and $P(\cdot)$ describes its production. Here P depends not on x at time τ but rather at a previous time $\tau - \tau_D$, and τ_D is regarded as the time-lag. Performing the scaling $t = \frac{\tau}{\tau_D}$, $u(t) = x(\tau_D t)$ and introducing $\varepsilon = \frac{1}{a\tau_D}$, $f(\cdot) := \frac{1}{a}P(\cdot)$, we obtain the scaled delayed recruitment/renewal equation

$$\varepsilon u'(t) = -u(t) + f(u(t-1)). \tag{2}$$

We remark that the perturbation parameter $\varepsilon \ll 1$ when $\tau_D \gg \frac{1}{a}$, which implies that delay differential equation (2) turns out to be singularly perturbed. For a singularly perturbed DDE, the solution has initial layers wherein the solution exhibits an exponential character and the potential for chaotic oscillations. We refer the reader to the papers [10,11] for the asymptotic expansion of the solution and the uniform exponential stability of singularly perturbed DDEs.

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The computational challenge arises from the smallness of the parameter ε and the singular perturbation nature of the problem. It is well known that standard discretization methods for solving singular perturbation problems are unstable and fail to give accurate results when the perturbation parameter ε is small. Therefore, it is important to develop suitable numerical methods to these problems, whose accuracy does not depend on the parameter value ε, that is, methods that are uniformly convergent with respect to the perturbation parameter.

There are essentially two strategies to design schemes which have small truncation errors inside the layer region(s). The first approach which forms the class of fitted-mesh methods consists in choosing a fine mesh in the layer regions (such as Shishkin meshes [12,13]). Amiraliyev and Erdogan [14] have given almost a first order uniformly convergent scheme by employing a backward difference operator on a non-uniform mesh which consists of the special piecewise uniform meshes on each time interval. Bawa, Lal and Kumar [15] proposed a first order uniformly convergent scheme which combines the implicit Trapezoidal scheme in the inner region with the backward difference operator in the outer region on a Shishkin mesh. The second approach is in the context of the exponential fitting methods in which the mesh remains uniform and the difference schemes reflect the qualitative behavior of the solution(s) inside the layer regions. For example, McCartin in [9] built second-order and fourth-order numerical schemes upon exponential fitting by replacing the function f in the variation-of-constants formula with Hermite interpolation polynomials.

The main purpose of this paper is to construct an efficient numerical scheme of the initial value problem for the delayed recruitment/renewal Eq. (2). In Section 2, we present the construction of a new family of exponential fitting Runge–Kutta methods. The uniform convergence analysis is given in Section 3. In Section 4, some numerical examples are conducted to illustrate the performance of our proposed methods. Finally, a conclusion is included in Section 5.

2. Exponential fitting Runge–Kutta methods of collocation type

We consider the discretization of the initial value problem for the delayed recruitment/renewal equation

$$\begin{cases} \varepsilon u'(t) = -u(t) + f(u(t-1)), \quad t \in (0, T], \\ u(t) = \varphi(t), \quad t \in [-1, 0], \end{cases} \tag{3}$$

where the initial value function $\varphi(t)$ is sufficiently differentiable for $t \in [-1, 0]$, f is sufficiently differentiable and $T \geq 1$ is a positive integer. We assume that there is a constant $L \geq 0$ such that $|f(v) - f(\tilde{v})| \leq L|v - \tilde{v}|$ for all $v, \tilde{v} \in \mathbb{R}$.

In order to employ the exponential fitting technique, we first rewrite it as

$$\frac{du}{dt} + \frac{1}{\varepsilon}u(t) = \frac{1}{\varepsilon}f(u(t-1)),$$

and then apply an integrating factor $e^{\frac{t}{\varepsilon}}$ to produce

$$\frac{d}{dt}\left[e^{\frac{t}{\varepsilon}}u(t)\right] = \frac{1}{\varepsilon}e^{\frac{t}{\varepsilon}}f(u(t-1)).$$

Integrating the above equation from t_{n-1} to t_n, we may obtain

$$e^{\frac{t_n}{\varepsilon}}u(t_n) - e^{\frac{t_{n-1}}{\varepsilon}}u(t_{n-1}) = \frac{1}{\varepsilon}\int_{t_{n-1}}^{t_n} e^{\frac{t}{\varepsilon}}f(u(t-1))dt,$$

which may be rearranged as

$$u(t_n) = e^{-\frac{h}{\varepsilon}}u(t_{n-1}) + \frac{1}{\varepsilon}\int_{t_{n-1}}^{t_n} e^{-\frac{t_n - t}{\varepsilon}}f(u(t-1))dt. \tag{4}$$

Let h be a given uniform step size such that $Nh = 1$ with some positive integer $N \geq 1$. Denote mesh points $t_i = ih,\ i = 0, 1, \ldots, u_n$ the approximation of $u(t_n)$ and $u_0 = \varphi(0)$.

The main idea behind exponential fitting integrators of collation type is to replace the function f in the variation-of-constants formula with a suitable polynomial. For $t = t_{n-1} + sh$, we deduce from (4) the following relation

$$u(t_{n-1} + h) = e^{-\frac{h}{\varepsilon}}u(t_{n-1}) + \frac{h}{\varepsilon}\int_0^1 e^{-\frac{h(1-s)}{\varepsilon}}f(u(t_{n-N-1} + sh))ds. \tag{5}$$

For integer $m \geq 1$, let $c_i \in [0, 1]$ $(i = 1, 2, \ldots, m)$ and $c_i \neq c_j$ when $i \neq j$ be distinct collocation nodes, and $U_i^{(n)}$ is a numerical approximation of $u(t_{n-1} + c_i h)$. We denote $U_i^{(n-N)} = \varphi(t_{n-N-1} + c_i h)$ for $n = 1, 2, \ldots, N$, $i = 1, 2, \ldots, m$, and $U_i^{(n-N)}$ is the numerical approximation of $u(t_{n-N-1} + c_i h)$ when $n > N$.

To obtain the collocation solution, we replace the function f in the integral by a collocation polynomial $p_{m-1}(s)$ and obtain the approximation

$$u_n = e^{-z}u_{n-1} + h\sum_{i=1}^{m} b_i(z)f(U_i^{(n-N)}), \tag{6}$$