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Worst case error bounds for the solution of uncertain Poisson equations with mixed boundary conditions

Tanveer Iqbal *, Arnold Neumaier

Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, 1090 Wien, Austria

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ABSTRACT

Given linear elliptic partial differential equations with mixed boundary conditions, with uncertain parameters constrained by inequalities, we show how to use finite element approximations to compute worst case a posteriori error bounds for linear response functionals determined by the solution. All discretization errors are taken into account.

Our bounds are based on the dual weighted residual (DWR) method of Becker and Rannacher (2001), and treat the uncertainties with the optimization approach described in Neumaier (2008).

We implemented the method for Poisson-like equations with an uncertain mass distribution and mixed Dirichlet/Neumann boundary conditions on arbitrary polygonal domains. To get the error bounds, we use a first order formulation whose solution with linear finite elements produces compatible piecewise linear approximations of the solution and its gradient. We need to solve nine related boundary value problems, from which we produce the bounds. No knowledge of domain-dependent a priori constants is necessary.

Contents

* Corresponding author.
E-mail addresses: iqbal.tanveer@univie.ac.at (T. Iqbal), arnold.neumaier@univie.ac.at (A. Neumaier).

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1. Introduction

In practical applications, partial differential equations represent an approximate model of the real life situation. In many applications, partial differential equations depend on parameters which are only approximately known. Modeling errors can also be accounted for by adding parameters (constants or functions) to the model and specifying the uncertainty in these parameters. Each parameter then represents a particular scenario from the set of possibilities. In practice, one can solve the equation for a particular scenario or for just a few scenarios. But one is interested in how the solution varies over the full set of allowed scenarios.

For partial differential equations, one needs not only to consider the uncertainty due to parameters but also the errors introduced by discretization. In a traditional sensitivity analysis, one usually neglects the discretization errors, and ignores higher order terms in the sensitivity analysis. These two types of errors may however significantly affect the validity of the resulting bounds.

The work by Nakao & Plum [1–3] presents rigorous error bounds for linear elliptic equations using interval analysis. It is mathematically rigorous and also accounts for roundoff errors and errors in the numerical integrations. The parameter-dependent case is also studied by Plum [4] and Yamamoto et al. [5]. The methods apply to Dirichlet boundary conditions, compute error estimation in global norms like the energy norm or L^2 norm, and assume the knowledge of domain-dependent a priori constants for key inequalities used. These are known only for a few domains.

In many applications, the error in the global norm does not provide useful bounds for the errors in the quantities of real physical interest. Here work exists only in the nonparametric case (no uncertainties). Bertsimas & Caramanis [6] present a method based on semidefinite optimization to get bounds on linear functionals of the solutions of elliptic equations with Dirichlet boundary conditions. In the work by Repin [7], a posteriori estimates have been derived with the help of duality theory from the calculus of variations. Work by the group of Peraire [8–10] used finite elements and a piecewise polynomial form of the coefficients to derive a posteriori error bounds for problems without uncertainty. Numerical integration errors and rounding errors are not taken into account.

No computable error bounds seem to be available in the case of mixed Dirichlet–Neumann boundary conditions treated in the present work.

Overview. In the present work, we discuss an approach that provides bounds on a linear response functional for a solution of mass-weighted Poisson equations with mixed boundary conditions on polygonal domains, with uncertain mass distribution. No a priori information is needed. We rigorously bound all discretization errors using new techniques, and bound the errors in the sensitivity analysis using the optimization approach outlined in Neumaier [11]. On the other hand, we shall assume that, compared to these errors, errors in the global optimization, errors in numerical integrations, and rounding errors can be neglected. For fully rigorous bounds, these would have to be taken into account, too.

We derive optimization-based error bounds for the discretization error, using a variant of the dual weighted residual (DWR) method by Becker & Rannacher [12]. For given uncertain parameters in the mass distribution m_θ, we compute the worst case error of a given linear response functional of the solution.

The first part of the present paper treats the problem in an abstract functional analytic setting. In Section 2, we discuss the spaces needed, and introduce the concept of a quasi-adjoint, used in Section 3 to derive abstract error bounds. Section 4 then discusses how we handle uncertainty in the differential equation.

The second part treats more specifically the mass-weighted Poisson equation. A first order formulation of the primal and adjoint equations is derived in Section 5. The θ-dependent operators M, N and E from the abstract theory are constructed for the mass-weighted Poisson equation in Section 6. The formulas for evaluating the dual norm of the residual of the adjoint problem are found in Section 7. Section 8 formulates an optimization problem whose solution defines suitable values of β, e, and $\mathbf{e}$ needed in the bounds.

The resulting algorithm was implemented in Matlab for the mass-weighted Poisson equation with mixed Dirichlet and Neumann boundary conditions on a polygonal domain in 2 dimensions. We illustrate the method with results for a particular example in Section 9.

2. Spaces and quasi-adjoint

Let U be a vector space and let V be a subspace of a Hilbert space $\overline{V}$. We write V^* for the dual space of V; thus $V \subseteq \overline{V} \subseteq V^*$. We write v^*v' for the inner product of $v, v' \in V$ and the bilinear pairing of $v \in V^*$ and $v' \in V$. Let $L \colon U \to \overline{V}$ be a linear operator mapping U into $\overline{V}$.

In the applications, U and V are spaces of locally differentiable functions. Thus V^* is a space of distributions obtained by differentiation of a square integrable function. L is composed of a first order differential operator and an associated boundary value mapping. We introduce the norm

$$\|v\|_V := \sqrt{v^*v} \quad \text{for } v \in \overline{V} \tag{1}$$

in $\overline{V}$. If L is injective then

$$\|u\|_U := \|Lu\|_V \tag{2}$$