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A modified fifth-order WENOZ method for hyperbolic conservation laws

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ABSTRACT

The paper analyses by Taylor series the several fifth-order of accuracy schemes for hyperbolic conservation laws: the classical WENOJS scheme Jiang and Shu (1996), the WENOM scheme Henrick et al. (2005), the WENOZ scheme Borges et al. (2008) and the scheme, called WENOε here, Arràndiga et al. (2011). The order of weights of these four schemes agreed to the optimal weights is presented in detail. Then three prerequisites are developed if one intends to improve the WENOJS scheme: the scheme arrives the 5th-order at critical points; the weights of scheme approximate the optimal weights with high-order accuracy when solution is smooth; the scheme should not introduce much oscillations intuitively in the vicinity of discontinuities. According to the prerequisites above, a new WENO scheme (MWENOZ) is devised which is similar to the WENOZ scheme. Finally, the method designed here is demonstrated robustly by applying it to 1D and 2D numerical simulations and its advantage compared with the WENOZ scheme seems more striking in 2D problems.

1. Introduction

In this paper, we are concerned with the improvement of 5th-order weighted essentially non-oscillatory schemes (WENO) for hyperbolic conservation laws [1], which are based on the well-known essentially non-oscillatory schemes (ENO) [2–5]. These high-order schemes were devised to capture the shocks (or contact discontinuities) without generating spurious oscillations essentially. The considerable success of WENO schemes owes to the adaptive combination of low-order stencils, which can keep high order of accuracy around smooth regions and ENO property near shocks. This nonlinear combination was firstly developed by Liu et al. in [6] for 3rd-order schemes, and then generalized by Jiang and Shu [1] for higher-order schemes.

In [7], however, Henrick et al. found that the order of accuracy of WENOJS at critical points was concealed if $\varepsilon = 10^{-6}$ occurring in the smoothness indicators of scheme. Actually, when the parameter ε was chosen to be much slighter, e.g. $\varepsilon = 10^{-40}$, WENOJS scheme only achieved the 3rd-order at critical points. They derived a very potent sufficient criteria for the 5th-order convergence. And a mapped function applied to the weights of WENOJS scheme was devised to satisfy this sufficient criteria and the optimal order was recovered at critical points. This scheme (WENOM) possesses significant improvement compared with WENOJS scheme regardless of smooth or discontinuous regions. In [8,9] the authors found that

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the weights of discontinued stencils in WENOM scheme were amplified by the mapped function, which produced instable solutions near large-gradient regions for long-time solutions, and a improved function was devised to overcome this defect. Although the WENOM scheme improves the WENOJS scheme with global optimal accuracy, the cost of computation is also increased accordingly. Another improved scheme (WENOZ) [10] modified the smoothness indicator in WENOJS scheme unlike the WENOM scheme. The advantage of WENOZ scheme is that it has almost same resolution (even higher) as WENOM but with almost the same computational cost as WENOJS. In addition, the WENOZ scheme cannot achieve the optimal order at critical points when $p = 1$ and recovers the optimal order when $p = 2$ but with slightly lower resolution (cf. [10]). Recently, Aràndiga et al. [11] exploited the special structure of WENOJS and gave compact formulae of smoothness indicators and nonlinear weights. Moreover, to get the optimal order of accuracy at critical points, the authors suggested the parameter ε should be chosen proportional to the square of mesh Δx^2. Then [12] presented necessary and sufficient condition that ε should satisfy when ε is regarded as a function of Δx.

By using the simple Taylor expansions, we analyse the smoothness indicators and nonlinear weights of the WENOJS, WENOM, WENOZ and WENOε schemes. The WENOJS scheme only achieves the 3rd-order of accuracy at critical points, and, more importantly, the nonlinear weights approximate the optimal weights with $\mathcal{O}(\Delta x^2)$. So, it is clear that, for smooth solutions, the WENOJS scheme is worse than the central scheme. For the WENOM scheme, the nonlinear weights approximate the optimal weights with $\mathcal{O}(\Delta x^6)$ and the order of accuracy at critical points is also recovered. Hence, the WENOM scheme apparently possesses higher resolution than the WENOJS scheme, which is also demonstrated numerically in [7]. Applying the same procedure to the WENOZ scheme, the approximated order of weights to optimal weights is $\mathcal{O}(\Delta x^5)$. So, the WENOZ and WENOM schemes should have similar behaviour. Finally, we also discuss the WENOε scheme which approximates the optimal weights with the same order as WENOJS. So, numerically, the WENOε scheme cannot show too much improvement in comparison with the WENOJS scheme despite of the recovery of order of accuracy at critical points. Furthermore, to get the optimal accuracy, the parameter ε is taken as a function of Δx. In conclusion, we present three prerequisites if one intends to improve the WENOJS scheme: the scheme arrives the 5th-order of accuracy at critical points; the weights of scheme approximate the optimal weights with high-order accuracy when solution is smooth; the scheme should not introduce much oscillations intuitively in the vicinity of discontinuities. According to the prerequisites above, a new WENO scheme (MWENOZ) is devised which is similar to the WENOZ scheme, but the variable ζ, like τ_5 in [10], is only composed of the second derivatives of interpolation polynomials.

The rest of this paper is organized as follows. Section 2 gives a brief review over the procedure of WENOJS, WENOM, WENOε schemes and analyses the order of accuracy of weights approximating the optimal weights. Section 3 reviews the WENOZ scheme and introduces the MWENOZ scheme. Section 4 applies the MWENOZ scheme to the 1D and 2D conservation laws and compares it with the WENOZ scheme.

2. The analysis of accuracy for WENOJS, WENOM and WENOε methods

In this section, we mainly review the several WENO methods [13,7,11] and analyse their order of accuracy with Taylor expansion.

2.1. Review of the WENOJS method

The WENOJS method was initially devised to solve the hyperbolic conservation laws

$$u_t + f(u)_x = 0, \quad a \le x \le b,\ t \ge 0, \tag{1}$$

on a uniform mesh

$$\Im:\ a = x_{1/2} < x_{3/2} < \cdots < x_{N-1/2} < x_{N+1/2} = b.$$

The cells I_i, cell centres x_i and length of cells Δx_i are respectively defined by

$$I_i = [x_{i-1/2}, x_{i+1/2}], \qquad x_i = (x_{i-1/2} + x_{i+1/2})/2, \qquad \Delta x_i = x_{i+1/2} - x_{i-1/2}.$$

By using the technique of method of lines, a conservative semi-discretized form of (1) can be written

$$\frac{du_i(t)}{dt} = -\frac{1}{\Delta x_i}\left[\hat{f}(x_{i+1/2}) - \hat{f}(x_{i-1/2})\right], \tag{2}$$

if the function $\hat{f}(x)$ satisfies

$$f(u_i) = \frac{1}{\Delta x_i}\int_{x_{i-1/2}}^{x_{i+1/2}} \hat{f}(x)dx. \tag{3}$$

Then the numerical fluxes $\hat{f}(x_{i\pm1/2})$ on the cell interfaces can be obtained by WENO reconstruction process.