Journal of Computational and Applied Mathematics 303 (2016) 119–139

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

A new compounding family of distributions: The generalized gamma power series distributions

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ARTICLE INFO

Article history:
Received 21 October 2014
Received in revised form 20 January 2016

Keywords:
Generalized gamma distribution
Generating function
Maximum likelihood estimator
Moment
Power series distribution

ABSTRACT

We propose a new four-parameter family of distributions by compounding the generalized gamma and power series distributions. The compounding procedure is based on the work by Marshall and Olkin (1997) and defines 76 sub-models. Further, it includes as special models the Weibull power series and exponential power series distributions. Some mathematical properties of the new family are studied including moments and generating function. Three special models are investigated in detail. Maximum likelihood estimation of the unknown parameters for complete sample is discussed. Two applications of the new models to real data are performed for illustrative purposes.

1. Introduction

The generalized gamma ($\mathcal{GG}$) distribution [1] is a well-known and established three-parameter distribution for modeling lifetime data and phenomenon with monotone failure rates. It is specially useful to fit bathtub hazard rate data (in addition to increasing, decreasing and unimodal shapes), thus overcoming the forms presented by the exponential, gamma and Rayleigh distributions for modeling this type of data. The $\mathcal{GG}$ distribution has been used in several research areas such as engineering, hydrology and survival analysis. However, in order to fit still more complex situations, a number of extensions have been proposed in recent years. For example, see the works by Cordeiro et al. [2], Ortega et al. [3], Cordeiro et al. [4] and the references therein.

A random variable T following the $\mathcal{GG}$ distribution with shape parameters $k > 0$, $\alpha > 0$ and scale parameter $\beta > 0$ has cumulative distribution function (cdf) given by

$$F_{\mathcal{GG}}(t) = \gamma_1\left(k, \left(\frac{t}{\beta}\right)^{\alpha}\right), \quad t > 0, \tag{1}$$

where $\gamma_1(k, z) = \gamma(k, z)/\Gamma(k)$ is the incomplete gamma function ratio and $\gamma(k, z) = \int_0^z \omega^{k-1} e^{-\omega} d\omega$ is the incomplete gamma function. The probability density function (pdf) corresponding to (1) is

$$f_{\mathcal{GG}}(t) = \frac{\alpha}{\beta\,\Gamma(k)}\left(\frac{t}{\beta}\right)^{k\alpha-1} \exp\left\{-\left(\frac{t}{\beta}\right)^{\alpha}\right\}, \quad t > 0. \tag{2}$$

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http://dx.doi.org/10.1016/j.cam.2016.02.033
0377-0427/

Stacy and Mihram [5] encountered some difficulties in developing maximum likelihood procedures and large sample inference for its parameters. On the other hand, Prentice [6] reparameterized it in such a way that the inference can be fairly easily handled. Lawless [7] by using Prentice's re-parametrization developed exact inference procedures concerning the quantiles and scale parameters from uncensored samples and DiCiccio [8] proposed approximate conditional inference methods for location and scale parameters. Recently, Huang and Hwang [9] presented a simple method for estimating the model parameters, using its characterization and moment estimation. An iterative estimation method for its parameters was implemented in S–PLUS by Gomes et al. [10]. Tadikamalla [11] proposed a simple rejection method for sampling directly from the $\mathcal{GG}$ distribution without generating gamma variates, but valid only for $\beta > 1$.

Our chief goal is to propose a new extension of the $\mathcal{GG}$ distribution by compounding the $\mathcal{GG}$ and power series ($\mathcal{PS}$) distributions. The generated class is called the *generalized gamma power series* ($\mathcal{GGPS}$) family. The compounding procedure follows the pioneering work of Marshall and Olkin [12]. In the same way, several classes of distributions were proposed by compounding some useful lifetime and $\mathcal{PS}$ distributions in the last few years. Chahkandi and Ganjali [13] defined the exponential power series ($\mathcal{EPS}$) class of distributions, which contains as special cases the exponential Poisson ($\mathcal{EP}$), exponential geometric ($\mathcal{EG}$) and exponential logarithmic ($\mathcal{EL}$) distributions. Morais and Barreto-Souza [14] defined the Weibull power series ($\mathcal{WPS}$) class which includes as sub-models the $\mathcal{EPS}$ distributions. The $\mathcal{WPS}$ distributions can have increasing, decreasing and upside down bathtub failure rate function. The generalized exponential power series ($\mathcal{GEPS}$) distributions were proposed by Mahmoudi and Jafari [15] following the same approach of Morais and Barreto-Souza [14]. Silva et al. [16] studied the extended Weibull power series ($\mathcal{EWPS}$) family, which includes as special models the $\mathcal{EPS}$ and $\mathcal{WPS}$ distributions. Bourguignon et al. [17] and Silva and Cordeiro [18] proposed the Birnbaum–Saunders power series ($\mathcal{BSPS}$) and Burr XII power series ($\mathcal{BXIIPS}$) classes of distributions, respectively.

The rest of the paper is organized as follows. In Section 2, we introduce and motivate the new family and present a useful representation for its density function. Section 3 gives an explicit expression for the $\mathcal{GGPS}$ moments. The moment generating function (mgf) is also derived in this section. We discuss in Section 4 three special models of the proposed family. Estimation of the parameters by maximum likelihood is addressed in Section 5. Section 6 gives two applications to real data to prove that the new family can be used quite effectively in analyzing lifetime data. Section 7 provides some conclusions.

2. The $\mathcal{GGPS}$ family of distributions

The new family of distributions is rather simple to be constructed following the same set-up carried out by Marshall and Olkin [12]. Given a discrete random variable N, let $X_1, \ldots, X_N$ be i.i.d. random variables having the $\mathcal{GG}$ distribution (1) with shape parameters $k, \alpha > 0$ and scale parameter $\beta > 0$, where N has a power series probability mass function (pmf) (truncated at zero) given by

$$p_n = P(N = n) = \frac{a_n\,\theta^n}{C(\theta)}, \quad n = 1, 2, \ldots. \tag{3}$$

The coefficients a_n's depend only on n and $C(\theta) = \sum_{n=1}^{\infty} a_n\,\theta^n$ (for $\theta > 0$) is assumed finite. It is important to remark that the probability class of distributions (3) has been considered in [19,20]. Table 1 lists some power series distributions (truncated at zero) defined by (3) such as the Poisson, logarithmic, geometric and binomial distributions. Let $X = \min\{T_i\}_{i=1}^{N}$. The conditional cumulative distribution of $X|N = n$ is given by

$$F_{X|N=n}(x) = 1 - \left[1 - \gamma_1\left(k, \left(\frac{x}{\beta}\right)^{\alpha}\right)\right]^n$$

i.e., $X|N = n$ has the exponentiated form of (1) with parameters n, k, α and β. So, we obtain

$$P(X \le x, N = n) = \frac{a_n\theta^n}{C(\theta)}\left\{1 - \left[1 - \gamma_1\left(k, \left(\frac{x}{\beta}\right)^{\alpha}\right)\right]^n\right\}, \quad x > 0,\ n = 1, 2, \ldots.$$

Then, the marginal cdf of X becomes

$$F_{\mathcal{GGPS}}(x; \theta, k, \alpha, \beta) = 1 - C(\theta)^{-1} C\left\{\theta\left[1 - \gamma_1\left(k, \left(\frac{x}{\beta}\right)^{\alpha}\right)\right]\right\}, \quad x > 0. \tag{4}$$

Eq. (4) is called the $\mathcal{GGPS}$ family of distributions.

Hereafter, a random variable X following (4) with parameters θ, k, α and β is denoted by $X \sim \mathcal{GGPS}(\theta, k, \alpha, \beta)$. Eq. (4) extends several other distributions which have been studied in the literature. The $\mathcal{EG}$ distribution [21] is obtained by taking $k = \alpha = 1$ and $C(\theta) = \theta(1-\theta)^{-1}$ with $\theta \in (0, 1)$. Further, for $k = \alpha = 1$, we obtain the $\mathcal{EP}$ [22] and $\mathcal{EL}$ [23] distributions by taking $C(\theta) = e^{\theta} - 1$, $\theta > 0$, and $C(\theta) = -\log(1-\theta)$, $\theta \in (0, 1)$, respectively. In the same way, for $k = 1$, we obtain the $\mathcal{WG}$ [24] and $\mathcal{WP}$ [25] distributions. The $\mathcal{EPS}$ distributions are obtained from (4) when $k = \alpha = 1$ for any $C(\theta)$ listed in Table 1 (see [13]). Finally, we obtain the $\mathcal{WPS}$ distributions from (4) by taking $k = 1$ for any $C(\theta)$ in Table 1 (see [14]). Some important sub-models of the $\mathcal{GG}$ distribution are listed in Table 2. This composition leads to 76 special models.