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A reliable incremental method of computing the limit load in deformation plasticity based on compliance: Continuous and discrete setting

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ABSTRACT

The aim of this paper is to introduce an enhanced incremental procedure that can be used for the numerical evaluation and reliable estimation of the limit load. A conventional incremental method of limit analysis is based on parametrization of the respective variational formulation by the loading parameter $\zeta \in (0, \zeta_{lim})$, where ζ_{lim} is generally unknown. The enhanced incremental procedure is operated in terms of an inverse mapping $\psi : \alpha \mapsto \zeta$ where the parameter α belongs to $(0, +\infty)$ and its physical meaning is work of applied forces at the equilibrium state. The function ψ is continuous, nondecreasing and its values tend to ζ_{lim} as $\alpha \to +\infty$. Reduction of the problem to a finite element subspace associated with a mesh $\mathcal{T}_h$ generates the discrete limit parameter $\zeta_{lim,h}$ and the discrete counterpart ψ_h to the function ψ. We prove pointwise convergence $\psi_h \to \psi$ and specify a class of yield functions for which $\zeta_{lim,h} \to \zeta_{lim}$. These convergence results enable to find reliable lower and upper bounds of ζ_{lim}. Numerical tests confirm computational efficiency of the suggested method.

1. Introduction

Elastic-perfectly plastic models belong among fundamental nonlinear models which are useful for estimation of yield strengths or failure zones in bodies caused by applied forces. Such models are mostly quasistatic (see, e.g., [1–3]) to catch the unloading phenomenon. Since we are only interested in monotone loading processes, this phenomenon can be neglected and the class of models based on the deformation theory of plasticity is adequate (see, e.g., [4,5,3,6,7]). The Hencky model associated with the von Mises yield criterion belongs to this class as well as other models with different yield conditions. Each model from this class leads to a static problem for a given load functional L representing the work of surface or volume forces. The problem can be formulated both in terms of stresses or displacements. These two approaches generate a couple of mutually dual problems.

The variational problem formulated in terms of stresses leads to minimization of a strictly convex, quadratic functional on the set of statically and plastically admissible stress fields. On the other hand, the stored energy functional appearing in the variational problem for displacements has only a linear growth at infinity with respect to the strain tensor or some

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components of this tensor. Existence of a finite limit load reflects specifics of this class of problems. Unlike other problems in continuum mechanics with superlinear growth of energy, exceeding of the limit load leads to absence of a solution satisfying the equilibrium equations and constitutive relations. Physically this means that under this load the body cannot exist as a consolidated object. Therefore, finding limit loads is an important problem in the theory of elasto-plastic materials and other close problems.

To introduce the limit load for the functional L the problem is usually parametrized at first. Instead of the fixed load, the set $\{\zeta L \mid \zeta \in \mathbb{R}_+\}$ of loads is considered. The limit value ζ_{lim} of the parameter is defined as a supremum of all $\zeta \geq 0$ for which the intersection of the sets of statically and plastically admissible stress fields is nonempty. In particular, no solution exists for the load ζL with $\zeta > \zeta_{lim}$.

There exist several approaches how to evaluate ζ_{lim}. The first type of methods is based on the use of a specific variational problem which characterizes directly the limit state. It can be formulated either in terms of displacements (kinematical approach) or in terms of stresses (static approach). Both are mutually dual [8,7]. As a computational method the static limit analysis has been used in [9], while the kinematic one in [10]. For example, the respective problem of kinematic limit analysis for the classical Hencky model with the von Mises condition reads as follows:

$$\zeta_{lim} = \inf_{\substack{v \in V,\, L(v)=1 \\ \mathrm{div}=0}} \int_\Omega |\varepsilon(v)|\, dx,$$

where $\mathbb{V}$ is a subspace of $H^1(\Omega; \mathbb{R}^3)$ of functions vanishing on the Dirichlet part of the boundary (see notation of Section 2). However, this problem is not simple for numerical analysis because it is related to a nondifferentiable functional and contains the divergence free constraint. The respective numerical approaches developed to overcome these difficulties often use saddle point formulations with augmented Lagrangians (see, e.g., [10,8]). Other methods use techniques developed for minimization of nondifferentiable functionals.

The classical approach uses incremental techniques to enlarge ζ up to its limit value [11,12]. The load increments have to be chosen adaptively since the value of ζ_{lim} is not known. The incremental limit analysis is usually combined with the standard finite element method and the resulting parametrized problem $(\mathcal{P}_h)_\zeta$ is then solved in terms of displacements. The main drawback of this approach is that the discrete limit value $\zeta_{lim,h}$ can overestimate ζ_{lim} and convergence of $\{\zeta_{lim,h}\}_h$ to ζ_{lim} is not guaranteed in general.

Besides ζ_{lim}, the incremental approach enables to detect other interesting thresholds on the loading path that represent global material response, namely, $\zeta_{e,h}$ – the end of elasticity and $\zeta_{prop,h}$ – the limit of proportionality. For $\zeta \leq \zeta_{e,h}$, the response is purely elastic (linear) and for $\zeta \in [\zeta_{prop,h}, \zeta_{lim,h}]$, the response is strongly nonlinear. To investigate global material response, it is necessary to introduce a quantity α depending on $\zeta < \zeta_{lim,h}$. For example, α can represent a computed displacement at a point in which the body response is the most sensitive on the applied load. Examples of such $\alpha - \zeta$ curves are introduced, e.g., in [1, Section 7,8].

In [13], the response parameter α has been introduced for the Hencky problem and the linear simplicial ($P1$) elements as follows: $\alpha = L(u_{h,\zeta})$ where $u_{h,\zeta}$ denotes a solution of $(\mathcal{P}_h)_\zeta$ for $\zeta < \zeta_{lim,h}$. This parameter is universal for any load and geometry. Moreover, there exists a function $\psi_h : \alpha \mapsto \zeta$ that is continuous, nondecreasing and satisfying $\psi_h(\alpha) \to \zeta_{lim,h}$ as $\alpha \to +\infty$. Further, for a given value of α, a minimization problem $(\mathcal{P}_h)^\alpha$ for the stored strain energy functional subject to the constraint $L(v) = \alpha$ has been derived. Its solution coincides with a solution to problem $(\mathcal{P}_h)_\zeta$ for $\zeta = \psi_h(\alpha)$ and thus the loading process can be controlled indirectly through the parameter α. Consequently, in [14], suitable numerical methods for both problems, $(\mathcal{P}_h)_\zeta$ and $(\mathcal{P}_h)^\alpha$, have been proposed and theoretically justified. Further, the load incremental methods controlled through ζ and α have been compared there.

The aim of this paper is to get reliable estimates of ζ_{lim} using the incremental procedure. To this end, we introduce a continuous, nondecreasing function $\psi : \mathbb{R}_+ \to (0, \zeta_{lim})$ satisfying $\psi(\alpha) \to \zeta_{lim}$ as $\alpha \to +\infty$. In comparison to [14,13], the function ψ is defined within a continuous setting of the problem and also for a general yield criterion. The derivation of ψ however is not straightforward owing to the fact that the primal formulation is not well-posed on classical Sobolev spaces. Therefore the dual formulation of the problem in terms of stresses will be used. Further, it is considered the discrete counterpart ψ_h of ψ within the $P1$ elements. In case of the von Mises yield criterion, the definition of ψ_h coincides with [14,13]. From the computational point of view, it is crucial to show that $\lim_{h\to 0_+} \psi_h(\alpha) = \psi(\alpha)$ for any $\alpha \geq 0$ and use the estimate $\psi(\alpha) \leq \zeta_{lim} \leq \zeta_{lim,h}$. We also specify a class of yield functions for which $\zeta_{lim,h} \to \zeta_{lim}$ holds.

The paper is organized as follows: In Section 2, we introduce basic notation, define elasto-plastic problems, and recall some results concerning properties of solutions. In Section 3, the loading parameters ζ and α are introduced. Then the function $\psi : \alpha \mapsto \zeta$ is constructed and its properties are established. In Section 4, we formulate problems in terms of stresses and displacements related to a prescribed value of α. Section 5 is devoted to standard finite element discretizations of the problems and to convergence analysis. Finally, in Section 6, we present two examples with different yield functions and compute lower and upper bounds of the limit load using the suggested incremental procedure.

2. Elastic-perfectly plastic problem based on the deformation theory of plasticity

We consider an elasto-plastic body occupying a bounded domain $\Omega \subseteq \mathbb{R}^3$ with Lipschitz boundary $\partial\Omega$. It is assumed that $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N$, where Γ_D and Γ_N are open and disjoint sets, Γ_D has a positive surface measure. Surface tractions of density f are applied on Γ_N and the body is subject to a volume force F.