Journal of Computational and Applied Mathematics 303 (2016) 171–188

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

High order time discretization for backward semi-Lagrangian methods

Francis Filbet [a,*], Charles Prouveur [b]

[a] Université Paul Sabatier & IUF, Institut de Mathématiques de Toulouse, 118 route de Narbonne, 31062 Toulouse Cedex 9, France

[b] Institut Camille Jordan, Université Claude Bernard, Lyon 1 43, boulevard du 11 novembre 1918 69 622 Villeurbanne Cedex, France

ARTICLE INFO

Article history:
Received 20 March 2015
Received in revised form 21 October 2015

MSC:
65M25
35Q83
82D10

Keywords:
High order time discretization
Semi-Lagrangian scheme
Vlasov–Poisson model
Guiding-centre model
Plasma physics

ABSTRACT

We introduce different high order time discretization schemes for backward semi-Lagrangian methods. These schemes are based on multi-step schemes like Adams–Moulton and Adams–Bashforth schemes combined with backward finite difference schemes. We apply these methods to transport equations for plasma physics applications and for the numerical simulation of instabilities in fluid mechanics. In the context of backward semi-Lagrangian methods, this time discretization strategy is particularly efficient and accurate when the spatial error discretization becomes negligible and allows to use large time steps.

1. Introduction

A model which can be used in many cases for the study of plasma as well as beam propagation is the Vlasov equation coupled with the Maxwell or Poisson equations to compute the self-consistent fields. It describes the evolution of a system of particles under the effects of external and self-consistent fields. The unknown $f(t, \mathbf{x}, \mathbf{v})$, depending on the time t, the position $\mathbf{x}$, and the velocity $\mathbf{v}$, represents the distribution of particles in phase space for each species with $(\mathbf{x}, \mathbf{v}) \in \mathbb{R}^d \times \mathbb{R}^d$, $d = 1, \ldots, 3$. Its behaviour is given by the Vlasov equation,

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + \mathbf{F}(t, \mathbf{x}, \mathbf{v}) \cdot \nabla_{\mathbf{v}} f = 0, \tag{1.1}$$

where the force field $F(t, \mathbf{x}, \mathbf{v})$ is coupled with the distribution function f giving a nonlinear system. We first define $\rho(t, \mathbf{x})$ the charge density and $\mathbf{J}(t, \mathbf{x})$ the current density which are given by

$$\rho(t, \mathbf{x}) = q \int_{\mathbb{R}^d} f(t, \mathbf{x}, \mathbf{v}) d\mathbf{v}, \qquad \mathbf{J}(t, \mathbf{x}) = q \int_{\mathbb{R}^d} \mathbf{v} f(t, \mathbf{x}, \mathbf{v}) d\mathbf{v},$$

where q is the elementary charge. For the Vlasov–Poisson model

$$\mathbf{F}(t, \mathbf{x}, \mathbf{v}) = \frac{q}{m} \mathbf{E}(t, \mathbf{x}), \quad \mathbf{E}(t, \mathbf{x}) = -\nabla_{\mathbf{x}} \phi(t, \mathbf{x}), \quad -\Delta_{\mathbf{x}} \phi = \frac{\rho}{\varepsilon_0}, \tag{1.2}$$

* Corresponding author.
E-mail addresses: francis.filbet@math.univ-toulouse.fr (F. Filbet), prouveur@math.univ-lyon1.fr (C. Prouveur).

http://dx.doi.org/10.1016/j.cam.2016.01.024
0377-0427/

where m represents the mass of one particle. On the other hand for the Vlasov–Maxwell model, we have

$$\mathbf{F}(t,\mathbf{x},\mathbf{v}) = \frac{q}{m}\left(\mathbf{E}(t,\mathbf{x}) + \mathbf{v}\times\mathbf{B}(t,\mathbf{x})\right),$$

and $\mathbf{E}$, $\mathbf{B}$ are solutions of the Maxwell equations

$$\begin{cases} \dfrac{\partial \mathbf{E}}{\partial t} - c^2\nabla\times\mathbf{B} = -\dfrac{\mathbf{J}}{\varepsilon_0}, \\ \dfrac{\partial \mathbf{B}}{\partial t} + \nabla\times\mathbf{E} = 0, \\ \nabla\cdot\mathbf{E} = \dfrac{\rho}{\varepsilon_0}, \qquad \nabla\cdot\mathbf{B} = 0, \end{cases}$$

with the compatibility condition

$$\frac{\partial \rho}{\partial t} + \mathrm{div}_{\mathbf{x}}\mathbf{J} = 0,$$

which is verified by the Vlasov equation solution.

Later on we will also consider the guiding centre model, which has been derived to describe the evolution of the charge density in a highly magnetized plasma in the transverse plane of a tokamak. This model is described as follows: we now consider the density ρ, solution to

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \mathbf{U}\cdot\nabla\rho = 0, \\ -\Delta\phi = \rho, \end{cases} \tag{1.3}$$

where the velocity $\mathbf{U} = (-\partial_y\phi, \partial_x\phi)$ is divergence free.

The numerical resolution of the Vlasov equation and related models is usually performed by Particle-In-Cell (PIC) methods which approximate the plasma by a finite number of particles. Trajectories of these particles are computed from characteristic curves given by the Vlasov equation, whereas self-consistent fields are computed on a mesh of the physical space. This method yields satisfying results with a relatively small number of particles but it is subject to fluctuations, due to the numerical noise, which are difficult to control.

To develop accurate and stable numerical techniques for plasma turbulence, numerical methods based on direct numerical simulation techniques have been developed. The Vlasov equation is discretized in phase space using either semi-Lagrangian [1–5], finite element [6], finite difference [7–10] or discontinuous Galerkin [11–14] schemes.

Among them, the semi-Lagrangian method consists of computing the distribution function at each grid point by following the characteristic curve ending there. Then to compute the value of the distribution function at the origin of the characteristic, a high order interpolation method is needed.

Concerning the time discretization of the backward differential system corresponding to the characteristic curve, it is often performed via a second order splitting scheme, but higher order techniques are also available [15]. This strategy is particularly interesting when each step of the splitting can be explicitly solved. However, it is not always possible to apply a splitting strategy [4], hence the time discretization technique is often limited to a leap-frog scheme or a second order predictor–corrector scheme. The aim of the present work is to propose and validate high order techniques to preserve the order of accuracy of the spatial discretization.

This paper is organized as follows: in Section 2 we recall the backward semi-Lagrangian method then in Section 3 we develop different time discretization schemes, based on multi-step methods. Both classical second order schemes and "new" third and fourth order schemes will be proposed. In Section 4, we present several numerical results. In the first, a two-dimensional transport equation with Gaussian initial data is investigated to recover the expected order of accuracy. In the second, we perform numerical simulations on the simplified paraxial Vlasov–Poisson model in two dimensions. The last three tests are applied to the guiding centre model to compare the performances of the schemes and to investigate the ability to recover stability results and the development of instabilities.

2. The backward semi-Lagrangian method

To construct a semi-Lagrangian method for (1.1) or (1.3), we consider a generic transport equation written in an advective form

$$\frac{\partial f}{\partial t} + \mathbf{A}(t,\mathbf{X})\cdot\nabla f = 0, \tag{2.1}$$

where $\mathbf{A} : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^d$, $d \geq 1$ and a set of mesh points $(\mathbf{x}_i)_{i\in\mathbb{Z}}$. The main feature of the backward semi-Lagrangian method is to compute an approximated solution to (2.1) by following the characteristics backward in time

$$\begin{cases} \dfrac{d\mathbf{X}}{dt} = \mathbf{A}(t,\mathbf{X}), \\ \mathbf{X}(t^{n+1}) = \mathbf{x}_i, \end{cases} \tag{2.2}$$