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An optimal cascadic multigrid method for the radiative transfer equation

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ABSTRACT

This paper presents a fast and optimal multigrid solver for the radiative transfer equation. A discrete-ordinate discontinuous-streamline diffusion method is employed to discretize the radiative transfer equation. Instead of utilizing conventional multigrid methods for spatial variables only, a spatial cascadic multigrid method and a full cascadic multigrid method are developed to achieve rapid convergence in iterative calculation. Preliminary analysis is also conducted, suggesting the optimal convergence rate. Numerical tests show a significant reduction of the computational time compared to conventional iterative methods for the radiative transfer equation.

1. Introduction

The radiative transfer equation (RTE), which describes the scattering and absorbing of radiation through a medium, plays an important role in a wide range of applications such as astrophysics [1], atmosphere and ocean [2–4], heat transfer [5], neutron transport and nuclear physics [6,7], and so on. Substantial research effort on the RTE began in the middle of last century. Today, research on the RTE remains to be a very active and important area, see e.g., the collections [8,9]. Recent development in biomedical engineering, such as radiotherapy dose calculations [10,11] and optical tomography [12–18], stimulates another giant wave of research in the area of radiative transfer calculations. To provide crucial information on the properties of the biological tissues, the radiative transfer cannot be ignored in a strongly spatially heterogeneous medium [19,20]. This spatial heterogeneity, on the other hand, escalates the complexity of the radiative transfer process, and results in high computational cost. Therefore, it is necessary to develop radiative transfer calculation schemes that are more suited for application to biomedical medium with a variety of tissues.

Due to the involvement of both integration and differentiation in the equation, as well as the high dimension of the problem, it is challenging to develop effective numerical schemes for solving the RTE. In the existing literature, the discretization schemes for the RTE are broadly categorized into two groups: deterministic (or explicit) and stochastic (or probabilistic). Stochastic methods [21–24] are usually employed for trajectory calculation, and are commonly considered as the golden-standard in terms of solution accuracy. However, that accuracy comes with a significant increase in both computational cost and memory requirement. In a deterministic approach, discretization of the RTE is usually carried out separately

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for the spatial and angular variables. Popular spatial discretizations include finite difference methods [25], finite element methods [26,27], finite volume methods [28], and so on. Various angular discretizations exist: discrete ordinate (S_n) methods [25,29], methods using spherical harmonics (P_n methods, [6]), etc. Some earlier references on this topic include [30,25], while a few more recent references are [31–33]. It is also worth mentioning that to deal with the strongly asymmetric scattering, such as highly forward-peaked scattering, various approximations to the RTE were developed (see, e.g., [34,35] and references therein).

The choice of a particular method depends on the conditions and purposes of the radiative transfer calculation, because deterministic and stochastic approaches have their own advantages and disadvantages. One of the advantages of the deterministic approach is that it is generally capable of providing the entire spatial and angular distribution of the radiation field at once. This means that all radiative quantities, such as the intensity of radiance in an arbitrary direction, the net flux or the flux in a particular direction, can be derived from the same basic calculation result. Another advantage is that it is free of statistical noise, which is inevitable in stochastic methods due to the finite number of photons employed in practice, and therefore deterministic methods are highly competent for quantitative estimation of the effects of changes in optical parameters on the radiation distribution. However, deterministic methods still require considerable computational resources and calculation times to obtain accurate results. Therefore, it is necessary to develop fast solver for the deterministic methods satisfying the conditions of both accuracy and feasibility required in various applications.

The multigrid method (see e.g. [36,37]) has been proved to be able to efficiently accelerate convergence and thus significantly reduce the computation time. For the radiative transfer equation, D. Balsara developed and studied a full approximation scheme (FAS) incorporated with the discrete ordinate method [38]; in [39,40], a parallel spatial/angular agglomeration multigrid method employing the FAS was developed to accelerate the finite-volume method for the radiative transfer equations; in [41], both FAS and full multigrid (FMG) method were applied to the spatial and angular variables for the steady-state or frequency domain radiative transfer equation.

The purpose of this paper is to study the performance of the cascadic multigrid method (CMG) for solving the RTE. As a "one-way multigrid" method, a distinctive feature of the CMG is the total absence of coarse grid corrections, which indicates that this type of multigrid method is easier to implement than the conventional multigrid methods. The CMG is not brand new, and has been proposed and analyzed for elliptic and parabolic partial differential equations in [42–47]. However, to the best of our knowledge, the CMG has not been applied to the RTE. In the proposed CMG, the discrete-ordinate discontinuous-streamline diffusion schemes developed in [48] are employed to discretize the RTE. Both spatial CMG method and full CMG method are considered. Different from the spatial cascadic multigrid (SCMG) method that uses the same angular partition for all multigrid levels, the angular and spatial cascadic multigrid (ASCMG) method employs coarser angular partition for coarser spatial mesh while keeps the finest angular partition on the finest level, and therefore can further accelerate the computation significantly. On the other hand, since different angular partitions are employed, the ASCMG method needs an additional interpolation operator involved. Our numerical results indicate that such interpolation errors can be easily swept out by the smoothing iterations. Moreover, the numerical experiments show that the proposed CMG methods lead to a significant reduction of the computational time compared to conventional iterative methods for the radiative transfer equation, and the ASCMG method achieves a faster convergence in comparison with the SCMG method.

The rest of this paper is organized as follows: In next section, we introduce the radiative transfer equation, the relevant notation, and recall the corresponding existence and uniqueness result. In Section 3, we shall briefly review a discrete-ordinate discontinuous-streamline diffusion method for the radiative transfer equation proposed in [48]. We then present details of the CMG for the radiative transfer equation and analyze the convergence properties in Section 4. In Section 5, several numerical examples are presented to illustrate the effectiveness and convergence properties of the proposed method. Concluding comments and remarks on future work are given in Section 6.

2. Radiative transfer equation

Let X be a bounded domain in $\mathbb{R}^d$ ($d = 2, 3$) with a smooth boundary ∂X. Denote by $\boldsymbol{n}(\boldsymbol{x})$ the unit outward normal at $\boldsymbol{x} \in \partial X$. Let Ω be the angular space, i.e., the unit circle in $\mathbb{R}^2$ for $d = 2$, or the unit sphere in $\mathbb{R}^3$ for $d = 3$. For each fixed direction $\boldsymbol{\omega} \in \Omega$, we introduce the following subsets of ∂X:

$$\partial X_{\omega,-} = \{\boldsymbol{x} \in \partial X : \boldsymbol{\omega} \cdot \boldsymbol{n}(x) < 0\}, \qquad \partial X_{\omega,+} = \{\boldsymbol{x} \in \partial X : \boldsymbol{\omega} \cdot \boldsymbol{n}(x) \geq 0\}.$$

Then, we define

$$\Gamma_- = \{(\boldsymbol{x}, \boldsymbol{\omega}) : \boldsymbol{x} \in \partial X_{\omega,-}, \boldsymbol{\omega} \in \Omega\}, \qquad \Gamma_+ = \{(\boldsymbol{x}, \boldsymbol{\omega}) : \boldsymbol{x} \in \partial X_{\omega,+}, \boldsymbol{\omega} \in \Omega\}$$

as the incoming and outgoing boundaries.

Let $u(\boldsymbol{x}, \boldsymbol{\omega})$ be the radiative intensity at position $\boldsymbol{x} \in X$ along direction $\boldsymbol{\omega} \in \Omega$. We define the integral operator S by

$$(Su)(\boldsymbol{x}, \boldsymbol{\omega}) = \int_\Omega g(\boldsymbol{x}, \boldsymbol{\omega} \cdot \hat{\boldsymbol{\omega}}) u(\boldsymbol{x}, \hat{\boldsymbol{\omega}})\, d\sigma(\hat{\boldsymbol{\omega}}), \tag{1}$$

with g a non-negative phase function satisfying the normalization condition

$$\int_\Omega g(\boldsymbol{x}, \boldsymbol{\omega} \cdot \hat{\boldsymbol{\omega}})\, d\sigma(\hat{\boldsymbol{\omega}}) = 1 \quad \forall \boldsymbol{x} \in X,\ \boldsymbol{\omega} \in \Omega. \tag{2}$$