Journal of Computational and Applied Mathematics 303 (2016) 229–248

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

A discrete commutator theory for the consistency and phase error analysis of semi-discrete C^0 finite element approximations to the linear transport equation

Travis Thompson *

Department of Mathematics, Texas A&M University, College Station, TX 77843, United States
Department of Computational and Applied Mathematics, Rice University, Houston, TX 77005, United States

ARTICLE INFO

Article history:
Received 26 August 2015
Received in revised form 20 February 2016

Keywords:
Finite elements
Transport equation
Consistency error
Mass lumping
Dispersion error
Superconvergence

ABSTRACT

A novel method, based on a discrete commutator, for the analysis of consistency error and phase relations for semi-discrete continuous finite element approximation of the one-way wave equation is presented. The technique generalizes to arbitrary dimension, accommodates the use of compatible quadratures, does not require the use of complex calculations, is applicable on non-uniform mesh geometries, and is especially useful when conventional Taylor series or Fourier approaches are intractable. Following the theory the analysis method is demonstrated for several test cases.

1. Introduction

When approximating time-dependent problems like the heat equation, the wave equation, or the time-dependent transport equation with piecewise polynomial finite elements, e.g. globally continuous functions whose element-wise restrictions are contained in the polynomial space $\mathbb{P}_k$ or $\mathbb{Q}_k$, it is common to lump the mass matrix since it facilitates the linear algebra, see e.g. [1–5]. This operation is often thought to be benign in terms of accuracy, since it can be proved, see [1–4,6], that it does not affect the overall approximation properties of the method provided the quadrature is accurate enough.

Although it is convenient numerically, mass lumping induces large dispersion errors when solving transport-like equations [7]. These errors show immediately when using non-smooth initial data since the high frequencies contained in non-smooth data do not travel at the right speed.

The theoretical analysis of the consistency error and phase error can be a challenging task; traditionally, this analysis is done in one space dimension and on uniform grids to facilitate the use of Taylor expansions and Fourier Analysis, respectively see e.g. [8–12]. The purpose of the current work is the introduction of a novel method of consistency and phase error analysis for $\mathbb{P}_k$ and $\mathbb{Q}_k$ finite element semi-discretizations in the context of the d-dimensional linear transport equation. The method generalizes aspects of both the Taylor series and Fourier Analyses, and is sufficiently flexible so as to handle consistent semi-discretizations as well as those produced by quadrature rules satisfying some compatability requirements on the nodal basis functions.

The paper is organized as follows: Section 2 introduces the semi-discrete approximation of the model transport problem in one-dimension, and the usual approach to determining consistency error and phase error recalled. Section 3 presents the discrete commutator theory: Section 3.1 introduces the model problem, defines the discrete residual, and defines the

* Correspondence to: Department of Computational and Applied Mathematics, Rice University, Houston, TX 77005, United States.
E-mail address: tthompson@rice.edu.

http://dx.doi.org/10.1016/j.cam.2016.02.042
0377-0427/

discrete commutator; Section 3.2 introduces a relaxation of the uniform mesh assumption; Section 3.3 establishes the interplay of the discrete commutator and Taylor series expansions; Section 3.4 gives the main results of the theoretical framework, intended for application to specific problems, and draws connections between discrete commutator analysis and results from Fourier analysis. The analysis technique is demonstrated in Section 4 where some familiar results are recovered in more generality, and without the need to compute any Taylor expansions or resort to trigonometric identities.

During the review process a recent, complementary work [13] was brought to light and establishes a result similar to that of Theorem 4.1 for consistent $\mathbb{Q}_k$ finite element discretizations, on uniform meshes. The analysis is carried out in one-dimension, and the tensor-product nature of the Q_k basis is used to lift the result to $\mathbb{R}^d$. We note that Theorem 4.1 applies to both $\mathbb{P}_k$ and $\mathbb{Q}_k$ finite elements on centro-symmetric domains and does not require complex calculations. The centro-symmetry assumption is further investigated in Remark 4.1.

2. One-dimensional analysis

The usual notion of consistency error is recalled in Section 2.1. Sections 2.2.1 and 2.2.2 review some standard tool for the analysis of the dispersive effects of mass lumping based on Taylor expansions and Section 2.3 establishes the dispersion relations, phase errors, and points per wavelength requirements for the consistent and mass lumped cases. These results will be useful later; the familiar reader may progress to Section 3.

2.1. One-dimensional problem and $\mathbb{P}_1$ approximation

Consider the following one-dimensional transport equation in the domain $\Omega = (a, b)$:

$$\partial_t u + \beta \partial_x u = 0, \qquad u(x, 0) = u_0(x), \qquad (x, t) \in \Omega \times \mathbb{R}_+, \tag{1}$$

equipped with periodic boundary conditions. The velocity field, $\beta \in \mathbb{R}$, is assumed constant. The solution to (1) is $u_0(x-\beta t)$, where we have identified u_0 and its periodic extension. To formulate the Galerkin approximation, the domain $\Omega = (a, b)$ is partitioned into N intervals $[x_i, x_{i+1}]$ for $i = 0, 1, \dots, N$. Let the quantity $h_{i+\frac{1}{2}} := |x_{i+1} - x_i|$ denote the diameter of the cell $[x_i, x_{i+1}]$. Let $\{\psi_1, \dots, \psi_N\}$ be the family composed of the continuous and piecewise linear Lagrange polynomials associated with the nodes $\{x_1, \dots, x_N\}$ and define $\mathbf{X}_h = \text{span}\{\psi_1, \dots, \psi_N\}$. Periodicity is accounted for by extending ψ_N over $[x_0, x_1]$ and by identifying $\psi_0|_{[x_0,x_1]}$ with $\psi_N|_{[x_0,x_1]}$.

Let $U_0(x) \in \mathbf{X}_h$ denote a reasonable approximation of $u_0(x)$; it could be for instance the Lagrange interpolant or the L^2-projection of u_0. A semi-discrete approximate solution to (1), say $U \in C^1([0, T]; \mathbf{X}_h)$, is constructed by using the Galerkin technique. This approximation satisfies $U(x, 0) = U_0(x)$ and Eq. (2) where $(f, g) = \int_\Omega fg \, \mathrm{d}x$ is the usual L^2 inner product.

$$b(U, v) := (\partial_t U + \beta \partial_x U, v) = 0, \quad \forall v \in \mathbf{X}_h, \ \forall t \geq 0. \tag{2}$$

Upon using the expansion $U(t, x) = \sum_{j=1}^{N} U_j(t)\psi_j(x)$, the above problem can be reformulated as follows:

$$0 = b(U, \psi_i) = \sum_{j=0}^{N} \left(m_{ij}\partial_t U_j(t) + a_{ij}U_j(t) \right), \tag{3}$$

where $m_{ij} = \int_\Omega \psi_j \psi_i \, \mathrm{d}x$, and $a_{ij} = \beta \int_\Omega \partial_x \psi_j \psi_i \, \mathrm{d}x$. The result is a system of ordinary differential equations

$$M\dot{U}(t) = -AU(t), \tag{4}$$

which can be solved using suitable methods.

2.2. Consistency error for the one-dimensional problem

The consistency error of approximation methods is traditionally estimated by inserting the exact solution into the discrete equations defining the approximation. This leads us to the following.

Definition 2.1. Let u denote the solution to (1). The consistency error at x_i for the Galerkin approximation (2) is defined to be

$$R_i[u](t) = \|\psi_i\|_{L^1(\Omega)}^{-1} \left(\sum_{j=1}^{N} \left(m_{ij}\partial_t u(t, x_j) + a_{ij}u(t, x_j) \right) \right). \tag{5}$$

The expression (5) is similar to the truncation error for finite difference methods. The usual approach is the substitution of the exact solution and the utilization of a Taylor expansion about x_i. We define the order of the truncation error for the