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Inverse spectral problem for pseudo-Jacobi matrices with partial spectral data

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ABSTRACT

In this paper, we study a special kind of matrices which are pseudo-Jacobi matrices. Four inverse eigenvalue problems are discussed, and the necessary and sufficient conditions are presented under which these problems are solvable. The corresponding algorithms and some examples are given. Numerical results demonstrate that these methods are of practical utility.

1. Introduction

Various works have considered the inverse eigenvalue problems. For example, Ole H. Hald proved that a real symmetric tridiagonal matrix with positive codiagonal elements is uniquely determined by its eigenvalues and the eigenvalues of the largest leading principal submatrix [1]. Hochstadt showed that a suitable set of prescribed eigenvalues defines a unique Jacobi matrix [2]. Ying-Hong Xu and Er-Xiong Jiang studied an inverse problem for periodic Jacobi matrices, and obtained a necessary and sufficient condition [3]. H.J. Landau given a nonconstructive proof of an existence theorem for the inverse eigenvalue problem for real symmetric Toeplitz (RST) matrices [4]. J. Peng et al. considered a special kind of matrices which are symmetric, all elements are equal to zero except for the first row, the first column and the diagonal elements, and the elements of the first row are positive except for the first one [5], etc.

The inverse eigenvalue problems have many applications in other subjects, e.g. in dynamics, the inverse eigenvalue problem for the spring-mass system, the inverse vibration problem for the discrete beam, the inverse quadratic eigenvalue problem in damped vibration system [6–11], etc. For the real symmetric case, there is a lot of research done in the last two decades (e.g., see [1,2,4,10,12–17] and the references therein).

As pointed out in [18,19], some inverse problems on Jacobi matrices appear in connection with the discretization of the one-dimensional Schrödinger equation

$$y''(x) + (\lambda - p(x))y(x) = 0, \quad x \in (-1, 1),$$

where $p(x)$ is a continuous function defined on $[-1, 1]$. The Hamiltonian H corresponding to the above equation is the Hermitian operator $H : L^2 \to L^2$ (the Hilbert space of square integrable real functions) defined by

$$y(x) \to \left(-\frac{d^2}{dx^2} + p(x)\right) y(x).$$

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In non-Hermitian quantum mechanics, the Hamiltonian H instead of being Hermitian is *pseudo*-Hermitian, i.e. $H = QH^*Q$, where Q is the signature operator (i.e., Q^2 is the identity). So, *pseudo*-Jacobi matrices are relevant in this context. The discretization and truncation, with appropriate boundary conditions, of the Schrödinger equation for non-Hermitian quantum mechanics with a symmetric potential $p(x) = p(-x)$ leads to the matrix

$$H = \begin{bmatrix} a_1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & a_2 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{n-1} & -1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & a_n \end{bmatrix},$$

where n is even and $a_1 = \bar{a}_n, a_2 = \bar{a}_{n-1}, \ldots, a_{\frac{n}{2}} = \bar{a}_{\frac{n}{2}+1}$. The matrix H is *pseudo*-Hermitian with respect to the orthogonal matrix

$$Q = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

It is of interest to deduce $p(x)$ from the knowledge of the spectrum of the operator.

The goal of this paper is the solution of inverse spectral problems for complex tridiagonal matrices of the form

$$A_n = \begin{bmatrix} a_1 & b_1 & 0 & \cdots & \cdots & 0 \\ \epsilon_1 b_1 & a_2 & b_2 & \cdots & \cdots & 0 \\ 0 & \epsilon_2 b_2 & a_3 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} & b_{n-1} \\ 0 & 0 & 0 & \cdots & \epsilon_{n-1} b_{n-1} & a_n \end{bmatrix}, \tag{1}$$

where $a_1, \ldots, a_n, \epsilon_1, \ldots, \epsilon_{n-1} \in \mathbb{C}$, $|\epsilon_j| = 1$, $j = 1, \ldots, n-1$, and $b_1, \ldots, b_{n-1} > 0$. The restriction on the positivity of all b_j is unimportant, since the sign of any $b_j, j = 1, \ldots, n-1$, can be changed by a unitary similarity using a diagonal matrix of the form $\mathrm{diag}(e^{i\varphi_1}, \ldots, e^{i\varphi_n})$. If $a_1, \ldots, a_n, \in R$ and $\epsilon_j = 1, j = 1, \ldots, n-1$, then A_n reduces to a Jacobi matrix

$$J_n = \begin{bmatrix} a_1 & b_1 & 0 & \cdots & \cdots & 0 \\ b_1 & a_2 & b_2 & \cdots & \cdots & 0 \\ 0 & b_2 & a_3 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} & b_{n-1} \\ 0 & 0 & 0 & \cdots & b_{n-1} & a_n \end{bmatrix}.$$

A matrix of the form (1) is called a pseudo-Jacobi matrix.

This paper is organized as follows. After addressing some known results of the inverse eigenvalue problems in Section 1, some properties of Jacobi matrix and pseudo-Jacobi matrix are stated in Section 2. The necessary and sufficient conditions for the solvability of inverse eigenvalue problems for some certain class of pseudo-Jacobi matrices are presented in Section 3. The corresponding numerical algorithms and some examples are given in Section 4. The conclusions are given in Section 5.

2. Properties of the pseudo-Jacobi matrices

Throughout this paper, we use A_n to denote the special kind of matrices defined as in (1), A_j to denote the $j \times j$ leading principal submatrix of A_n. Let $\varphi_j(\lambda) = det(\lambda I - A_j)$. For convenience, let $\varphi_0(\lambda) = 1$. For a given $n \times n$ matrix T, we use $T_{j,n}$ to denote the submatrix of T with rows and columns from j to n, clearly, $T_{1,n} = T$. In addition, we denote $\varphi_{j,n} = det(\lambda I - T_{j,n})$.

Lemma 1. *For pseudo-Jacobi matrix A_n, the sequence $\{\varphi_j(\lambda)\}$ satisfies the recurrence relation*

$$\varphi_j(\lambda) = (\lambda - a_j)\varphi_{j-1}(\lambda) - \epsilon_{j-1} b_{j-1}^2 \varphi_{j-2}(\lambda), \quad j = 2, 3, \ldots, n. \tag{2}$$

Proof. It is easy to verify by expanding the determinant. $\square$

Lemma 2. *For pseudo-Jacobi matrix A_n, the sequence $\{\varphi_j(\lambda)\}$ satisfies*

(i) *there is no zero of $\varphi_0(\lambda)$;*
(ii) *there is no common root for $\varphi_j(\lambda)$ and $\varphi_{j-1}(\lambda)$.*