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A class of triangular splitting methods for saddle point problems ☆

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ABSTRACT

In this paper, we study a class of efficient iterative algorithms for the large sparse nonsingular saddle point problems based on the upper and lower triangular (ULT) splitting of the coefficient matrix. We call these algorithms ULT methods. First, the ULT algorithm is established and the characteristic of eigenvalues of the iteration matrix of these new methods is analyzed. Then we give the sufficient and necessary conditions for the convergence of these ULT methods. Moreover, the optimal iteration parameters and the corresponding convergence factors for some special cases of the ULT methods are presented. Numerical experiments on a few model problems are presented to support the theoretical results and examine the numerical effectiveness of these new methods.

1. Introduction

We consider the iterative solution of the large sparse saddle point problems of the form

$$\begin{pmatrix} A & B \\ B^{\top} & O \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}, \tag{1.1}$$

where $A \in R^{m\times m}$ is a nonsingular matrix, $B \in R^{m\times n}$ is a matrix of full column rank ($m \geq n$), and $p \in R^m$, $q \in R^n$ are two given vectors. Systems of linear equations with the form (1.1) are called saddle point problems which arise in many application areas. Generally speaking, both A and B are large and sparse.

Many practical problems arising from scientific computing and engineering applications may require the solution of systems of linear equations of the form (1.1). For example, computational fluid dynamics [1–4], optimal control [5], the finite element approximation for solving the Navier–Stokes equation [6], constrained optimization [7], the Lagrange-type methods for constrained nonconvex optimization problems [8], weighted least-squares problems [9–11], electronic networks [1,2,12], the nonlinear primal–dual methods for the Euler–Lagrange equations from total variation-based image restorations [13,14], the domain decomposition methods of the Helmholtz equations [15], computer graphics and so on. The system of

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linear equation (1.1) is also termed as an augmented system, or an equilibrium system, or a Karush–Kuhn–Tucker (KKT) system [16–18].

Because the matrices A and B are usually large and sparse, iterative methods become more attractive than direct methods for solving the saddle point problem (1.1), although the direct methods play an important role in the form of preconditioners embedded in an iterative framework. If B is rank-deficient in the linear system (1.1), then the coefficient matrix is singular, and (1.1) is called a singular saddle point problem. In the case that the coefficient matrix of Eq. (1.1) is positive real, Zhang [19] et al. studied the inexact Uzawa method, which covers the Uzawa method [20], the preconditioned Uzawa method [3], and the parameterized Uzawa (PU) method [21]. They also proved the semi-convergence result under restrictions by verifying two necessary and sufficient conditions. Sufficient conditions for the semi-convergence of several Uzawa-type methods were also provided in [19]. Moreover, many other techniques have been proposed for solving the rank-deficient saddle point problems. For example, in [22] Bai considered the HSS iteration method (first proposed in [23]) for singular linear systems, and they also gave a necessary and sufficient condition for guaranteeing the semi-convergence of this iteration method. In [24] Bai et al. proposed the preconditioned Hermitian and skew-Hermitian splitting (PHSS) methods for non-Hermitian positive semidefinite linear systems, and then the convergence property of the PHSS method was given in [25]. Furthermore, the scholars also presented preconditioned minimum residual (PMINRES) method [26] and preconditioned conjugate gradient (PCG) method [27] for solving the singular saddle point problems, and a preconditioned AHSS iteration method for singular saddle point problems was presented in [28].

In the case that the coefficient matrix of Eq. (1.1) is nonsingular, i.e. $\text{rank}(B) = n$, many efficient iterative methods as well as their numerical properties have been studied in the literature. For example, Uzawa-type methods [20,21,29–32], matrix splitting iterative methods [33–38], relaxation iterative methods [39,40], RPCG [41,42], Krylov subspace iterative methods, (often using block-diagonal, block-tridiagonal, constraint, SOR and HSS preconditioners) [41,43–46], iterative null space methods [47,48] and the references therein. Santos et al. [49], and Santos and Yuan [50] studied preconditioned iterative method for solving system (1.1) with $A = I$. Yuan [51] presented preconditioned conjugate gradient methods for solving general augmented systems like (1.1) where A can be symmetric and positive semidefinite and B can be rank deficient. In [20] Arrow et al. presented the classical Uzawa method which is a very efficient iteration method for solving the saddle point problems. After that, the improvement and development of the classical Uzawa method had been put forward by many other authors. Elman et al. [3] proposed the inexact Uzawa method which was further analyzed in [10]. Bai et al. [21] presented an efficient parameterized Uzawa (PU) method for solving the nonsingular saddle point problems, also termed as the generalized successive overrelaxation (GSOR) method, which includes the SOR-like method [38] and the classical Uzawa method [20] as special cases. Zheng and Ma [31] presented a class of accelerated Uzawa (AU) algorithms for solving the large sparse nonsingular saddle point problems by making use of the extrapolation technique which is based on the eigenvalues of the iterative matrix. Theoretical analysis and numerical experiments have shown that the AU method is feasible and effective for solving the saddle point problems. Moreover, in [34] Zheng and Ma proposed a new SOR-Like (NSOR-Like) method which has three parameters, and the iteration method can be applied to the nonsingular saddle point problems as well as the singular cases.

In this paper, we establish an upper and lower triangular (ULT) splitting (ULTS) method for solving the large sparse nonsingular saddle point problem (1.1). The ULT method is based on the upper and lower triangular (ULT) splitting of the coefficient matrix of linear system (1.1). The convergence of this new algorithm is studied. Also, the optimal iteration parameters and the corresponding convergence factors for some special cases of the ULT method are presented. And the numerical experiments further confirm the effectiveness of our new method.

The organization of this paper is as follows: in Section 2, we present the upper and lower triangular (ULT) splitting method to solve saddle point problems. In Section 3, analysis of the convergence property of this new method is given. Moreover, some numerical experiments are given to show the efficiency of the ULT method in Section 4. Finally, some conclusions are proposed in Section 5.

The following notations will be used throughout this paper. We denote the identity matrix and the 0-matrix by I and O, respectively. For a matrix $C \in R^{m\times n}$, we denote the transpose of C by $C^\top$, and the rank of matrix C is denoted as $\text{rank}(C)$. $\lambda_{\max}$ and $\lambda_{\min}$ denote the largest and smallest eigenvalues of the corresponding matrix, respectively. Moreover, the spectral radius of C is denoted by $\rho(C)$. $\|\cdot\|_2$ denotes the l_2 norm of the corresponding vector.

2. The ULT method

In this section, we will present the ULT iteration method for solving the saddle point problem (1.1). For the sake of simplicity, we rewrite linear system (1.1) as

$$\begin{pmatrix} A & B \\ -B^\top & O \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ -q \end{pmatrix}, \tag{2.1}$$

where $A \in R^{m\times m}$ is nonsingular and $B \in R^{m\times n}$ is a matrix of full column rank, i.e. $\text{rank}(B) = n$. Denote

$$\mathcal{W} = \begin{pmatrix} A & B \\ -B^\top & O \end{pmatrix}, \qquad z = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad g = \begin{pmatrix} p \\ -q \end{pmatrix},$$