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Nearest matrix with prescribed eigenvalues and its applications

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ABSTRACT

Consider an $n \times n$ matrix A and a set Λ consisting of $k \leq n$ prescribed complex numbers. Lippert (2010) in a challenging article, studied geometrically the spectral norm distance from A to the set of matrices whose spectra included specified set Λ and constructed a perturbation matrix Δ with minimum spectral norm such that $A+\Delta$ had Λ in its spectrum. This paper presents an easy practical computational method for constructing the optimal perturbation Δ by improving and extending the methodology, necessary definitions and lemmas of previous related works. Also, some conceivable applications of this issue are provided.

1. Introduction

Let A be an $n \times n$ complex matrix and let L be the set of complex $n \times n$ matrices that have $\lambda \in \mathbb{C}$ as a prescribed multiple eigenvalue. In 1999, Malyshev [1] obtained the following formula for the spectral norm distance from A to L:

$$\min_{B \in L} \|A - B\|_2 = \max_{\gamma \geq 0} s_{2n-1}\left(\begin{bmatrix} A - \lambda I & \gamma I_n \\ 0 & A - \lambda I \end{bmatrix}\right),$$

where $\|\cdot\|_2$ denotes the spectral matrix norm and $s_1(\cdot) \geq s_2(\cdot) \geq s_3(\cdot) \geq \cdots$ are the singular values of the corresponding matrix in nonincreasing order. Also he constructed a perturbation, Δ, of matrix A such that $A + \Delta$ belonged to the L and Δ was the optimal perturbation of the matrix A. Malyshev's work can be considered as a solution to Wilkinson's problem, that is, the computation of the distance from a matrix $A \in \mathbb{C}^{n \times n}$ which has only simple eigenvalues, to the set of $n \times n$ matrices with multiple eigenvalues. Wilkinson introduced this distance in [2] and some bounds were computed for it by Ruhe [3], Wilkinson [4–7] and Demmel [8].

However, in a non-generic case, if A is a normal matrix then Malyshev's formula is not directly applicable. Ikramov and Nazari [9] showed this point and they obtained an extension of Malyshev's formula for normal matrices. Furthermore, Malyshev's formula was extended by them [10] for the case of a spectral norm distance from A to matrices with a prescribed triple eigenvalue. In 2011, under some conditions, a perturbation Δ of matrix A was constructed by Mengi [11] such that

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Δ had minimum spectral norm and $A+\Delta$ belonged to the set of matrices that had a prescribed eigenvalue of prespecified algebraic multiplicity. Moreover, Malyshev's work also was extended by Lippert [12] and Gracia [13]. They computed a spectral norm distance from A to the matrices with two prescribed eigenvalues.

Recently, Lippert [14] provided a geometric basis for the results obtained in [13,12] and also, he introduced a geometric motivation for the smallest perturbation in the spectral norm such that the perturbed matrix had some given eigenvalues.

Denote by $\mathcal{M}_k$ the set of $n \times n$ matrices that have $k \le n$ prescribed eigenvalues. This article, motivated by the above spectrum updating problems, studies and considers the spectral norm distance from A to $\mathcal{M}_k$. In particular, it describes a clear and intelligible computational technique for construction of Δ having minimum spectral norm and satisfying $A+\Delta \in \mathcal{M}_k$. Expanding and improving the methodology used in [1,13,12,10,11] for the case of $k \le n$ fixed eigenvalues, and presenting a general solution for the matrix nearness problem in an evident computational manner are the main goals considered herein. This paper is also intended to provide definitions and proofs in a broad way and give sufficient conditions with the intention of encompassing the aforesaid works. On the other hand, in [10,14, Section 5] and [12, Section 6] it was mentioned that the optimal perturbations are not always computable for the case of fixing three or more distinct eigenvalues. This paper, provides two assumptions such that the optimal perturbation is always computable when these assumptions hold. It is noticeable that if one or both conditions are not satisfied then still $A+\Delta \in \mathcal{M}_k$, but Δ has not necessary minimum spectral norm. In this case we can have some lower bounds and $\|\Delta\|_2$ as an upper bound for the spectral norm distance from A to $A+\Delta \in \mathcal{M}_k$. Meanwhile, a selection of possible applications of this topic is considered.

Note that if, in a special case, A is a normal matrix, i.e., $A^*A = AA^*$, then we cannot use the method described in this paper for the computation of the perturbation, immediately. In this case, by following the analysis performed in [15,9,16] one can derive a refinement of our results for the case of normal matrices. Therefore, throughout of this paper, it is assumed that A is not a normal matrix. Suppose now that an $n \times n$ matrix A and a set of complex numbers $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ in which $k \le n$, are given. Now for

$$\gamma = \{\gamma_{1,1}, \gamma_{2,1}, \ldots, \gamma_{k-1,1}, \gamma_{1,2}, \gamma_{2,2}, \ldots, \gamma_{k-2,2}, \ldots, \gamma_{1,k-1}\} \in \mathbb{C}^{\frac{k(k-1)}{2}},$$

define the $nk \times nk$ upper triangular matrix $Q_A(\gamma)$ as

$$Q_A(\gamma) = \begin{bmatrix} B_1 & \gamma_{1,1}I_n & \gamma_{1,2}I_n & \cdots & \gamma_{1,k-1}I_n \\ 0 & B_2 & \gamma_{2,1}I_n & \cdots & \gamma_{2,k-2}I_n \\ \vdots & \ddots & B_3 & \ddots & \vdots \\ & & & \ddots & \gamma_{k-1,1}I_n \\ 0 & \cdots & & 0 & B_k \end{bmatrix}_{nk\times nk}, \tag{1}$$

where $\gamma_{i,1}$, $(i = 1, \ldots, k-1)$ are real variables and $B_j = A - \lambda_j I_n$, $(j = 1, \ldots, k)$.

Clearly, $Q_A(\gamma)$ can be assumed as a matrix function of complex variables $\gamma_{i,j}$ for $i = 1, 2, \ldots, k-1$, $j = 1, 2, \ldots, k-i$. Hereafter, for the sake of simplicity, the positive integer $nk-(k-1)$ is denoted by κ. Assume that the spectral norm distance from A to $\mathcal{M}_k$ is denoted by $\rho_2(A, \mathcal{M}_k)$, i.e,

$$\rho_2(A, \mathcal{M}_k) = \|\Delta\|_2 = \min_{M\in\mathcal{M}_k} \|A - M\|_2 .$$

Generally, this paper follows the plan including two main phases: Computing a lower bound, say α, for $\|\Delta\|_2$ and next constructing a perturbation matrix Δ such that $\|\Delta\|_2 = \alpha$, (or as close as possible to α) and $A+\Delta \in \mathcal{M}_k$. Due to this, some lower bounds for 2-norm of the optimal perturbation Δ are obtained in the next section. In Section 3, selected properties of κth singular value of $Q_A(\gamma)$, i.e., $s_\kappa\,(Q_A(\gamma))$, and its corresponding singular vectors are studied. These properties will be used in Section 4, with the purpose of computing the optimal perturbation $\|\Delta\|_2$, having minimum 2-norm and satisfying $A+\Delta \in \mathcal{M}_k$. Finally, in Section 6, some numerical examples and imaginable implementation of the topic of the matrix nearness problem are given to illustrate the validation and application of the presented method.

2. Lower bounds for the optimal perturbation

Let us begin by considering $s_\kappa\,(Q_A(\gamma))$ which is the κth singular value of $Q_A(\gamma)$. First, note that $s_\kappa\,(Q_A(\gamma))$ is a continuous function of variable γ. Also, if we define the unitary matrix U of the form

$$U = \begin{bmatrix} I_n & & & & 0 \\ & -I_n & & & \\ & & I_n & & \\ & & & \ddots & \\ 0 & & & & (-1)^{k-1}I_n \end{bmatrix}_{nk\times nk},$$