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Portfolio selection problem with Value-at-Risk constraints under non-extensive statistical mechanics

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ABSTRACT

The optimal portfolio selection problem is a major issue in the financial field in which the process of asset prices is usually modeled by a Wiener process. That is, the return distribution of the asset is normal. However, several empirical results have shown that the return distribution of the asset has the characteristics of fat tails and aiguilles and is not normal. In this work, we propose an optimal portfolio selection model with a Value-at-Risk (VaR) constraint in which the process of asset prices is modeled by the non-extensive statistical mechanics instead of the classical Wiener process. The model can describe the characteristics of fat tails and aiguilles of returns. Using the dynamic programming principle, we derive a Hamilton–Jacobi–Bellman (HJB) equation. Then, employing the Lagrange multiplier method, we obtain closed-form solutions for the case of logarithmic utility. Moreover, the empirical results show that the price process can more accurately fit the empirical distribution of returns than the familiar Wiener process. In addition, as the time increases, the constraint becomes binding. That is, to control the risk the agent reduces the proportion of the wealth invested in the risky asset. Furthermore, at the same confidence level, the agent reduces the proportion of the wealth invested in the risky asset more quickly under our model than under the model based on the Wiener process. This can give investors a good decision-making reference.

1. Introduction

The optimal portfolio selection problem is a major issue in finance. In 1952, Markowitz firstly proposed a single-period mean–variance approach for optimal portfolio selection problem [1]. Li and Ng studied a multi-period mean–variance optimal portfolio selection problem using the embedding technique [2]. Zhou and Li studied a continuous-time mean–variance optimal portfolio selection problem using the linear quadratic approach [3]. Yao et al. investigated a continuous-time mean–variance optimal portfolio selection problem using the Lagrange duality and the dynamic programming approach and obtained closed-form solutions for the optimal portfolio strategies [4]. However, in the mean–variance model, the risk is defined as the variation of portfolio returns, which implies that both negative yields and positive yields are regarded as a risk. Recently, to overcome that defect, several approaches measured the downside risk have been proposed, including the lower partial moment method [5,6], the semi-variance method [7,8], the conditional tail expectation method [9,10] and the

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Value-at-Risk method [11,12]. Since the Value-at-Risk approach is easily understood, it has become extremely popular in the practice of risk management. It is the maximum loss of a portfolio at a certain probability level over a given time horizon.

However, in the above models, the price processes of risky assets were modeled by the classical Wiener process. That is, the returns of risky assets follow normal distributions. While, time-series data usually has nonlinear and long-range dependent features [13,14]. Several empirical results have shown that the return distributions of risky assets have the characteristics of fat-tails and aiguilles and are not normal [15–17]. To more accurately fit the returns of risky assets, several models have been developed, such as discontinuous models with Poisson jumps [18–20] or based on levy processes [21–24]. Moreover, in 1988, famous physicist Tsallis proposed the non-extensive statistical theory [25], which has been developed greatly in physics. Lately, it has also increasingly drawn attention from finance researchers. Several results have shown that the power-law characteristics of Tsallis distributions can fit the distributions of certain financial quantities very well. For example, Ramos [26] Kozuki and Fuchikami [27] found that the distributions of foreign exchange rates have the power-law characteristics and they can be modeled by Tsallis distributions. Michael [28] found that a Tsallis distribution of index $q = 1.43$ can better fit the returns of Standard and Poor's 500 (S&P) index than the normal distribution. Borland [29,30] firstly used the non-extensive statistical mechanics to model the changes of stock prices and obtained closed-form solutions for European options, which generalized the classical Black–Scholes model.

In this paper, we propose a portfolio selection model in the framework of the non-extensive statistical mechanics and impose a Value-at-Risk constraint on it. The price process of the model can characterize fat-tails and aiguilles of the returns, while the familiar Wiener process cannot do that. Moreover, the model imposed the Value-at-Risk constraint is more applicable to risk management.

The paper is organized as follows. In Section 2, we give the model of stock prices and the definition of the Value-at-Risk constraint. In Section 3, we derive the Hamilton–Jacobi–Bellman (HJB) equation using the dynamic programming principle and present the Lagrange multiplier method to address the Value-at-Risk constraint problem. Moreover, the closed-form solutions for maximized expected logarithmic utility are given. In Section 4, the empirical results are presented and discussed. In the final section, we summarize the paper.

2. Market model and Value-at-Risk (VaR) constraint

In our model, the market consists of $p + 1$ assets. One is a risk-free bond whose price process $S_0(t)$ satisfies

$$dS_0(t) = rS_0(t)dt \tag{1}$$

where r is a positive risk-free rate. The other p risky assets are stocks whose price processes satisfy

$$dS_i(t) = S_i(t)\left(\mu_i dt + \sum_{j=1}^{p} \sigma_{i,j} d\Omega_j(t)\right), \quad 0 \le t < \infty,\ i = 1, 2, \ldots, p \tag{2}$$

where

$$d\Omega_j(t) = P(\Omega_j, t)^{\frac{1-q_j}{2}} dB_j(t). \tag{3}$$

The process $\{B(t)\} = \{B_j(t),\ j = 1, 2, \ldots, p\}$ is a p-dimensional standard Gaussian noise process. $P_q(\Omega)$ is the Tsallis distribution of index q as follow

$$P(\Omega, t) = \frac{1}{z(t)}\left(1 - \beta(t)(1-q)\Omega^2\right)^{\frac{1}{1-q}} \tag{4}$$

where

$$z(t) = ((2-q)(3-q)ct)^{\frac{1}{3-q}} \tag{5}$$

$$\beta(t) = c^{\frac{1-q}{3-q}}((2-q)(3-q)t)^{\frac{-2}{3-q}} \tag{6}$$

$$c = \frac{\pi}{q-1}\frac{\Gamma^2\left(\frac{1}{q-1} - \frac{1}{2}\right)}{\Gamma^2\left(\frac{1}{q-1}\right)}. \tag{7}$$

In the limit $q \to 1$, Eq. (4) recovers a Gaussian distribution. For $q > 1$, it exhibits power law tails. In Eq. (2), $\mu = (\mu_1, \mu_2, \ldots, \mu_p)^T$ is an R^p valued mean of returns and $\sigma = \{\sigma_{i,j},\ i, j, = 1, 2, \ldots, p\}$ is a $p \times p$-matrix valued variance covariance.

In our model, the agent is allowed to consume and the consumption process is denoted $\{c(t)\}$ which is positive. Let $\{\pi(t), c(t)\} = \{\pi_1(t), \pi_2(t), \ldots, \pi_p(t), c(t)\}$ be a control process, namely, portfolio-proportion process. The components $\pi(t)$ are the proportions of the agent's wealth invested in stocks. $c(t)$ is his consumption rate. Then, we can give the wealth