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A new run-up algorithm based on local high-order analytic expansions

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HIGHLIGHTS

- New asymptotic solutions in the vicinity of the shoreline are derived.
- A novel run-up algorithm is proposed based on more accurate knowledge of the shoreline dynamics.
- This algorithm is tested against the known analytical solutions and experimental data.

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ABSTRACT

The practically important problem of the wave run-up is studied in this article in the framework of Nonlinear Shallow Water Equations (NSWE). The main novelty consists in the usage of high order local asymptotic analytical solutions in the vicinity of the shoreline. Namely, we use the analytical techniques introduced by S. Kovalevskaya and the analogy with the compressible gas dynamics (*i.e.* gas outflow problem into the vacuum). Our run-up algorithm covers all the possible cases of the wave slope on the shoreline and it incorporates the new analytical information in order to determine the shoreline motion to higher accuracy. The application of this algorithm is illustrated in several important practical examples. Finally, the simulation results are compared with the well-known analytical and experimental predictions.

1. Introduction

The wave run-up problem has been attracting a lot of attention of the hydrodynamicists, coastal engineers and applied mathematicians because of its obvious practical importance for the assessment of inundation maps and mitigation of natural

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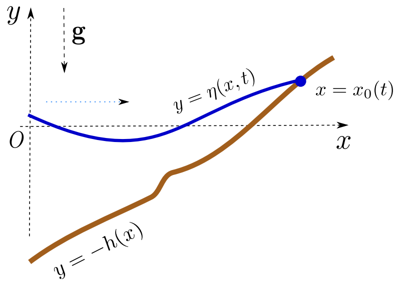

Fig. 1. Sketch of the physical domain, where $y = -h(x)$ is the bottom shape, $y = \eta(x, t)$ is the free surface and $x = x_0(t)$ is the moving shoreline. The horizontal blue dotted arrow shows the wave propagation direction.

hazards [1]. Most often this problem has been approached in the context of Nonlinear Shallow Water Equations (NSWE) which was successfully validated several times [2,3].

Various linearized theories have been applied to estimate the wave run-up [4]. Perhaps, the most outstanding method is the widely-known Carrier–Greenspan hodograph transformation which allows to transform the NSWE into a linear wave equation [5]. Later, this method was extended to solve also the Boundary Value Problem (BVP) for the NSWE [6]. The general conclusion to which several authors have converged is that the linear theory is able to predict correctly the maximal wave run-up [7]. However, this technique has at least one serious shortcoming since it is limited only to constant sloping beaches. Therefore, the usage of numerical techniques seems to be unavoidable [8].

The first tentatives to simulate numerically the run-up date back to the mid 70s [9]. Various techniques have been tested ranging from the analytical mappings to a fixed computational domain to the application of moving grids. The most widely employed technique consists in replacing the dry area by a thin water layer of negligible height (see [10] among the others). Perhaps, the first *modern* numerical run-up algorithm was proposed by Hibberd and Peregrine (1979) [11].

The run-up algorithm proposed in the present study is based on two ingredients. The first trick consists in discretizing the fluid domain solely which allows us to use judiciously the grid points only where they are needed in contrast to shock-capturing schemes where the whole computational domain (wet $\bigcup$ dry areas) is discretized. The other ingredient consists in using a high order asymptotic expansion near the moving shoreline point. To the lowest order these solutions can be identified with the so-called *shoreline Riemann problem* [12–14]. However, our asymptotic solutions are valid not only for flat, but also for general bottoms. In some particular cases they can provide us with the *exact* solution when it is a polynomial function in time. This novel analytical tool allows us to make a zoom on the NSWE solutions structure locally in time in a wider class of physical situations. This algorithm is completely unknown outside the Russian literature [15], which justifies the present publication. Moreover, in the present study we focus on the motion of the shoreline which is used in the run-up algorithm, instead of obtaining the solutions local in time in the whole domain.

The key idea to derive and apprehend these asymptotics lies in the deep analogy between the NSWE and the compressible Euler equations for an ideal polytropic gas. Another similarity consists in the analogy between the wave run-up (wetting/drying) process and the compressible gas outflow into the vacuum. In this case, the shoreline can be identified with the vacuum boundary. Consequently, one can hope to transpose the powerful analytical methods of the compressible gas dynamics [16,17] to NSWE. This program will be accomplished hereinbelow.

This paper is organized as follows. In Section 2 the governing equations are presented and some of their basic properties are discussed. Section 3 presents a novel high order asymptotic solution in the vicinity of the shoreline point. The numerical run-up algorithm based on this solution is described in Section 4 and some numerical results are presented in Section 5. Finally, the main conclusions and perspectives of this study are outlined in Section 6.

2. Mathematical model

Consider an ideal and incompressible fluid bounded from below by the absolutely rigid solid bottom (given by the equation $y = -h(x)$) and by the free surface on the top (given by $y = \eta(x, t)$). The sketch of the physical domain and the description of the chosen coordinate system are given in Fig. 1. The Cartesian coordinate system xOy is chosen such that the horizontal axis $y = 0$ coincides with the mean water level (*i.e.* the undisturbed position). The vector $\mathbf{g} = (0, -g)$ denotes the gravity acceleration. The fluid domain is bounded on the right by a sloping beach and $x = x_0(t)$ denotes the instantaneous position of the shoreline.

If we make an additional assumption on the shallowness of the gravity wave propagation (*i.e.* the characteristic wavelength is much bigger than the mean water depth), then it can be shown that the fluid flow is described by the classical Nonlinear Shallow Water Equations (NSWE) [18,12]:

$$\frac{\partial \boldsymbol{v}}{\partial t} + \frac{\partial \boldsymbol{f}(\boldsymbol{v})}{\partial x} = \boldsymbol{S}(\boldsymbol{v}, x), \quad x < x_0(t),\ t > 0. \tag{2.1}$$