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The airfoil equation on near disjoint intervals: Approximate models and polynomial solutions

Leandro Farina [a,c,*], Marcos R.S. Ferreira [a], Victor Péron [b]

[a] Instituto de Matematica, Universidade Federal do Rio Grande do Sul, Porto Alegre, RS, Brazil
[b] Team Magique-3D, INRIA Bordeaux-Sud-Ouest, LMAP CNRS UMR 5142 & Université de Pau et des Pays de l'Adour, France
[c] BCAM (Basque Center for Applied Mathematics), Bilbao, Basque Country, Spain

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ABSTRACT

The airfoil equation is considered over two disjoint intervals. Assuming the distance between the intervals is small an approximate solution is found and relationships between this approximation and the solution of the classical airfoil equation are obtained. Numerical results show the convergence of the solution of the original problem to the approximation. Polynomial solutions for an approximate model are obtained and a spectral method for the generalized airfoil equation on near disjoint intervals is proposed.

1. Introduction

Several works deal with the airfoil equation over an interval and its generalizations, see e.g. [1–4]. In its original form, this equation can be written as

$$\frac{1}{\pi}\int_{-1}^{1}\frac{f(t)}{x-t}\mathrm{d}t = g(x), \quad x\in(-1,1), \tag{1.1}$$

and it models the flow over an infinitely thin airfoil without viscosity and in the bidimensional case [5]. Here f is the local circulation density and it can be shown to be equal to the difference in induced velocity over the upper and lower side of the airfoil. The right-hand side g is given by the product of 2π, the onset velocity and the slope of airfoil which in case of a flat plate is nothing but the product of 2π, the onset velocity and the angle of incidence. The case when g is an odd function includes for instance an airfoil with zero angle of incidence and parabolic camber distribution. Applications in elasticity and surface wave scattering are also described by this equation [6].

In this work, we consider the airfoil equation with a Cauchy type of singularity and set in close disjoint intervals (2.1); that is, in two intervals with a small aperture of size $2\varepsilon > 0$. Tricomi [2] derived analytical solutions for the airfoil equation set in two disjoint intervals. Dutta and Banerjea [7] presented this solution in a slightly different form and have used it to deduce a solution of the associated hypersingular integral equation.

We investigate formally and numerically the asymptotic behavior for the solution f_ε of the airfoil equation when the small parameter ε tends to zero. We first identify the formal limit of f_ε as ε tends to zero: The background model f_0 is given

* Corresponding author at: Instituto de Matematica, Universidade Federal do Rio Grande do Sul, Porto Alegre, RS, Brazil.
E-mail addresses: farina@mat.ufrgs.br (L. Farina), marcos.rondiney@ufrgs.br (M.R.S. Ferreira), victor.peron@inria.fr (V. Péron).

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by a Cauchy singular integral. We compare the formal limit with the solution of the airfoil equation set in an interval $(-1, 1)$, in Section 3.2. We exhibit the first two terms of an expansion in power series of ε for particular analytical solutions f_ε and we specify a functional framework in weighted Sobolev spaces, Section 3.3. We illustrate numerically convergence results of f_ε towards f_0 in ε^2 in this framework, Section 4. We then find polynomials as solutions for the approximate model and we exhibit a spectral method to solve a generalized airfoil equation set on close disjoint intervals, in Section 5.

2. The mathematical model

We consider the airfoil equation with Cauchy type singularity and set in the union of two disjoint intervals with a small hole of size $2\varepsilon > 0$, $G_\varepsilon = (-1, -\varepsilon) \cup (\varepsilon, 1)$

$$\frac{1}{\pi}\int_{-1}^{-\varepsilon} \frac{f_\varepsilon(t)}{x-t}\mathrm{d}t + \frac{1}{\pi}\int_{\varepsilon}^{1} \frac{f_\varepsilon(t)}{x-t}\mathrm{d}t = -\psi(x), \quad x \in G_\varepsilon. \tag{2.1}$$

Here, f_ε is the unknown and ψ represents the data of the problem. We assume that ψ is Hölder continuous for $x \in G_\varepsilon$, $\psi \in \mathcal{C}^{0,\alpha}(G_\varepsilon)$.

Let $\varepsilon > 0$ be a fixed parameter. It is possible to solve the airfoil equation (2.1) by the method of Tricomi [2] (see also e.g. [7]). The solution writes

$$f_\varepsilon(x) = \begin{cases} \dfrac{1}{\pi R_\varepsilon(x)}[C_{1,\varepsilon} + C_{2,\varepsilon}x + \Psi_\varepsilon(x)], & x \in (-1, -\varepsilon) \\ -\dfrac{1}{\pi R_\varepsilon(x)}[C_{1,\varepsilon} + C_{2,\varepsilon}x + \Psi_\varepsilon(x)], & x \in (\varepsilon, 1) \end{cases} \tag{2.2}$$

where Ψ is given by the Cauchy integrals

$$\Psi_\varepsilon(x) = \int_{-1}^{-\varepsilon} \frac{\psi(t)R_\varepsilon(t)}{x-t}\mathrm{d}t - \int_{\varepsilon}^{1} \frac{\psi(t)R_\varepsilon(t)}{x-t}\mathrm{d}t. \tag{2.3}$$

$C_{1,\varepsilon}$ and $C_{2,\varepsilon}$ are two arbitrary constants, and $R_\varepsilon(x) = \sqrt{(1-x^2)(x^2-\varepsilon^2)}$, $x \in G_\varepsilon$.

We will take the constants $C_{1,\varepsilon}$ and $C_{2,\varepsilon}$ as zero and will call the associated solution as the *null-circulation solution*. The physical meaning and motivation for choosing homogeneous constants is due to the fact that this case relates to the solution of the Airfoil equation on an interval when the circulation around the airfoil is zero. We will see more details of this in Section 3.2.

3. Convergence results

In this section, we present convergence results and approximate models for the solution f_ε of (2.1) when ε tends to 0 (Proposition 3.2). We derive the background model f_0 which is nothing but the pointwise limit of f_ε as ε tends to 0. We compare the background model f_0 with the solution of the airfoil equation over an interval (1.1) (Proposition 3.4). Finally, we exhibit the first two terms of an asymptotic expansion of f_ε in power series of ε when the right hand side is a Chebyshev polynomial.

3.1. Asymptotic behavior. Background model

We first investigate the behavior of the solution of the airfoil equation (2.1) when ε goes to 0.

Lemma 3.1. *Let $x \in G_\varepsilon$ and $\psi \in \mathcal{C}^{0,\alpha}(-1, 1)$. Define the function g_ε on $G_\varepsilon \setminus \{x\}$ as*

$$g_\varepsilon(t) = \frac{\psi(t)R_\varepsilon(t)}{x-t}\mathbb{1}_{(-1,-\varepsilon)}(t) - \frac{\psi(t)R_\varepsilon(t)}{x-t}\mathbb{1}_{(\varepsilon,1)}(t). \tag{3.1}$$

Then, g_ε satisfies the pointwise convergence result: for a.e. $t \in (-1, 1)$,

$$g_\varepsilon(t) \longrightarrow -\frac{\sqrt{1-t^2}}{x-t}t\psi(t)\mathbb{1}_{(-1,1)}(t) \quad \text{as } \varepsilon \to 0. \tag{3.2}$$

Proof. For all $t \in G_\varepsilon \setminus \{x\}$, $g_\varepsilon(t) = \frac{\psi(t)R_\varepsilon(t)}{x-t}\left(\mathbb{1}_{(-1,-\varepsilon)} - \mathbb{1}_{(\varepsilon,1)}\right)(t)$. Observe that $R_\varepsilon(t) \longrightarrow |t|\sqrt{1-t^2}$ as $\varepsilon \to 0$ for all $t \in G_\varepsilon$ and $\left(\mathbb{1}_{(-1,-\varepsilon)} - \mathbb{1}_{(\varepsilon,1)}\right)(t) \longrightarrow -\,\mathrm{sgn}(t)\mathbb{1}_{(-1,1)}(t)$ as $\varepsilon \to 0$ for a.e. $t \in (-1, 1)$. As a consequence, there holds for a.e. $t \in (-1, 1)$

$$R_\varepsilon(t)\left(\mathbb{1}_{(-1,-\varepsilon)} - \mathbb{1}_{(\varepsilon,1)}\right)(t) \longrightarrow -\sqrt{1-t^2}\,t\mathbb{1}_{(-1,1)}(t) \quad \text{as } \varepsilon \to 0.$$

We infer the convergence result (3.2). ∎