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Spectral properties of a class of matrix splitting preconditioners for saddle point problems☆

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ABSTRACT

Based on the accelerated Hermitian and skew-Hermitian splitting iteration scheme (Bai and Golub, 2007), we propose a new two-parameter matrix splitting preconditioner in this paper. Spectral properties of the preconditioned matrix are analyzed in detail. Furthermore, based on this preconditioner, an improved version of matrix splitting preconditioner is presented and analyzed. Finally, performance of the preconditioners is compared by using GMRES(m) as an iterative solver on linear systems arising from the discretization of Stokes and Navier–Stokes equations.

1. Introduction

In this paper, we consider saddle point systems of linear equations in the form of

$$\begin{pmatrix} A & B^T \\ -B & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ -b \end{pmatrix} \quad \text{or} \quad \mathcal{A}u = f, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times n}$ and $m \leq n$. We assume that A is positive real, i.e., $H = \frac{1}{2}(A + A^T)$, the symmetric part of A is symmetric positive definite. We assume that the matrix $B \in \mathbb{R}^{m\times n}$ has full row rank and $\mathcal{N}(H) \cap \mathcal{N}(B) = 0$, where $\mathcal{N}(H)$ represents the null space of matrix H, i.e., $\mathcal{N}(H) = \{x \in \mathbb{R}^n \mid Hx = 0\}$. These assumptions guarantee that the linear system (1) has a unique solution [1]. Alternatively, the saddle point problem (1) can also be written in an equivalent form

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}. \tag{2}$$

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The system of linear equations (2) has the symmetric-like form, whereas it may be highly indefinite. As a comparison, the eigenvalues of the coefficient matrix of problem (1) are distributed in the right half plane [1]. The property is favorable to the convergence of iterative solvers [2]. Therefore, we focus on (1) in this paper.

The saddle point problems of form (1) frequently arise in many applications, such as constraint optimization, computational fluid dynamics, electromagnetism and so on. A number of numerical methods have been proposed for solving saddle point problems, including the Schur complement reduction methods, the null space methods, Uzawa-type methods, GSOR and PIU iterative methods [3], and the restrictively preconditioned conjugate gradient methods [4,3,5], see [1] and references therein. The coefficient matrix $\mathcal{A}$ is usually large and ill-conditioned, which makes convergence extremely slow when iterative methods are used to solve the problems. It is well recognized that preconditioning is crucial to make the iterative methods having fast convergence rate [2]. In recent years, considerable efforts have been made in developing efficient preconditioners, and several types of preconditioning techniques have been proposed, for example the block diagonal (triangular) preconditioners, constraint preconditioners [6,3], and some matrix splitting preconditioners [7,8]. Particularly, based on the HSS iteration, Benzi and Golub proposed an HSS preconditioner for generalized saddle point problems, see [3,9,10,7,11] for details. The eigenvalue bounds of the preconditioned matrix are provided in [12,3,1] for the HSS preconditioner. In [13], Pan, Ng and Bai developed an efficient matrix splitting preconditioner for solving (1), which was based on the matrix splitting iteration proposed in [14]. In this paper, a relaxed matrix splitting preconditioner is proposed based on the matrix splitting proposed in [14] and a two-parameter (α and β) splitting iteration newly proposed by Bai and Golub [9]. The spectral properties of the preconditioned matrix are analyzed in detail. In particular, we show that when $\beta > 0, \alpha \to 0_+$, the eigenvalues of the preconditioned matrix will be clustered around two points, one point is $(0, 0)$ and another point is $(2, 0)$. Based on the spectrum analysis and idea of optimized approximation scheme proposed in [7], we construct an improved matrix splitting preconditioner. The spectrum properties of the preconditioned matrix by the improved preconditioner are analyzed and compared with that of the two-parameter preconditioner. Both preconditioners are tested on a variety of problems arising from the discretization of Stokes, Navier–Stokes equations and distributed control problems. The numerical results indicate that the improved preconditioner is very efficient and robust with respect to relaxation parameters.

The reminder of this paper is organized as follows. In Section 2, we propose a two-parameter preconditioner induced from a matrix splitting. In Section 3, we analyze the spectral properties of the preconditioned matrix by the two-parameter preconditioner. In Section 4, we propose an improved matrix splitting preconditioner, and analyze the spectrum properties of the preconditioned matrix. Finally, numerical experiments are performed to illustrate the efficiency of both preconditioners in Section 5.

Throughout the paper, we use superscript ^{T} to denote the conjugate transpose of a vector or matrix, and $\|\cdot\|$ to denote both the Euclidean vector norm and the subordinate spectral matrix norm. We denote the identity matrix of order n by I_n.

2. Two-parameter matrix splitting preconditioner

Observe that the coefficient matrix $\mathcal{A}$ can be split into two parts as follows:

$$\mathcal{A} = \mathcal{T} + \mathcal{S},$$

where

$$\mathcal{T} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{S} = \begin{pmatrix} 0 & B^T \\ -B & 0 \end{pmatrix}. \tag{3}$$

Let Λ be a diagonal matrix

$$\Lambda = \begin{pmatrix} \alpha I_n & 0 \\ 0 & \beta I_m \end{pmatrix}$$

where α, β are positive parameters. Based on the coefficient matrix $\mathcal{A}$, we have the following matrix splitting,

$$\mathcal{A} = (\Lambda + \mathcal{T}) - (\Lambda - \mathcal{S}) = (\Lambda + \mathcal{S}) - (\Lambda - \mathcal{T}).$$

Similar to the two-parameter splitting iteration method introduced in [9], the following alternative iteration formula follows,

$$\begin{cases} (\Lambda + \mathcal{T})u^{k+\frac{1}{2}} = (\Lambda - \mathcal{S})u^k + f, \\ (\Lambda + \mathcal{S})u^{k+1} = (\Lambda - \mathcal{T})u^{k+\frac{1}{2}} + f, \end{cases} \tag{4}$$

where u^0 is an initial guess. The iteration scheme (4) can be regarded as a generalization of the BASI iteration method proposed in [12].