Journal of Computational and Applied Mathematics 298 (2016) 152–166

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

On moment based density approximations for aggregate losses

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ARTICLE INFO

Article history:
Received 24 August 2015
Received in revised form 6 November 2015

Keywords:
Gamma distribution
Moments
Weibull distribution

ABSTRACT

Jin et al. (2015) proposed a novel moments based approximation based on the gamma distribution for the compound sum of independent and identical random variables. They illustrated their approximation using six examples. Here, we revisit four of their examples. We show that moments based approximations based on simpler distributions can be good competitors. We also show that the moments based approximations are more accurate than truncated versions of the exact distribution of the compound sum.

1. Introduction

Suppose $X_1, X_2, \ldots, X_N$ are insurance claims made over a fixed period. Then the total claim over the period is

$$S = \sum_{i=1}^{N} X_i.$$

Closed form expressions for the distribution of S are hard to find. Jin et al. [1] developed quite a novel moments based approximation based on the gamma distribution. By means of six examples, they showed that their approximations perform well. The six examples were: (i) $X_1, X_2, \ldots, X_N$ are independent and identical gamma random variables and N is a Poisson random variable independent of $X_1, X_2, \ldots, X_N$; (ii) $X_1, X_2, \ldots, X_N$ are independent and identical inverse Gaussian random variables and N is a Poisson random variable independent of $X_1, X_2, \ldots, X_N$; (iii) $X_1, X_2, \ldots, X_N$ are independent and identical gamma random variables and N is a negative binomial random variable independent of $X_1, X_2, \ldots, X_N$; (iv) $X_1, X_2, \ldots, X_N$ are independent and identical inverse Gaussian random variables and N is a negative binomial random variable independent of $X_1, X_2, \ldots, X_N$; (v) $X_1, X_2, \ldots, X_N$ are independent and identical Pareto random variables and N is a Poisson random variable independent of $X_1, X_2, \ldots, X_N$; (vi) $X_1, X_2, \ldots, X_N$ are independent and identical Pareto random variables and N is a negative binomial random variable independent of $X_1, X_2, \ldots, X_N$.

While showing that their approximations perform well, Jin et al. [1] considered only one set of parameter values for each of the six examples. It is not clear if their approximations perform well for all possible values of the parameters. It is also not clear if other moments based approximations can perform better when parameters values are different. To investigate this, we revisit the first four of their examples and see how the gamma approximation compares to four other moments based approximations for a wide range of parameter values. The four other approximations are based on exponentiated exponential [2], Weibull [3], inverse Gaussian [4] and lognormal distributions. We have not considered the last two of Jin et al. [1]'s examples because the sum of independent and identical Pareto random variables does not have a closed form

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http://dx.doi.org/10.1016/j.cam.2015.11.048
0377-0427/

distribution function. The sum of independent and identical gamma random variables and the sum of independent and identical inverse Gaussian random variables do have closed form distribution functions.

The gamma distribution has an exponential tail while the inverse Gaussian distribution has a heavy tail. Both distributions are popular models in actuarial science, in particular for claims data. See Sections 5.2 and 5.4 of Kleiber and Kotz [5] for details and references of applications.

The contents of this paper are organized as follows. The gamma approximation due to Jin et al. [1] and the four other moments based approximations are discussed in Section 2. The performances of these approximations for the first four examples in Jin et al. [1] are compared in Section 3. Some concluding remarks are given in Section 4.

2. Approximations

The gamma approximation for the probability density function of S proposed in Jin et al. [1, Section 2] is

$$f_{\text{Approx},gamma,t}(s) = \frac{s^{\alpha-1}\exp(-s/\theta)}{\theta^{\alpha}\Gamma(\alpha)} \sum_{r=0}^{t} c_r s^r, \tag{1}$$

where

$$\alpha = \frac{\mu^2(1)}{\mu(2)-\mu^2(1)}, \qquad \theta = \frac{\mu(2)-\mu^2(1)}{\mu(1)},$$

and

$$\begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_t \end{pmatrix} = \begin{pmatrix} m_0 & \cdots & m_t \\ m_1 & \cdots & m_{t+1} \\ \vdots & \ddots & \vdots \\ m_t & \cdots & m_{2t} \end{pmatrix}^{-1} \begin{pmatrix} \mu(0) \\ \mu(1) \\ \vdots \\ \mu(t) \end{pmatrix}, \tag{2}$$

where

$$\mu(a) = [1-\Pr(N=0)]^{-1} \frac{d^a}{dt^a} E\left[\{E\left[\exp(tX)\right]\}^N\right]\Big|_{t=0} \tag{3}$$

and $m_i = \theta^i \Gamma(\alpha+i)/\Gamma(\alpha)$. The inner expectation in (3) is with respect to X. The outer expectation in (3) is with respect to N. As described in Jin et al. [1, Section 2], t is a positive integer determining the degree of accuracy of (1).

Following the method in Jin et al. [1, Section 2], other approximations for the distributions of S can be derived by replacing the two-parameter gamma probability density function, $s^{\alpha-1}\exp(-s/\theta)/\left[\theta^{\alpha}\Gamma(\alpha)\right]$, by some other two-parameter probability density function. We take that to be one of the two-parameter exponentiated exponential distribution, two-parameter Weibull distribution, two-parameter inverse Gaussian distribution or the two-parameter lognormal distribution. These and the gamma distribution are among the most popular two-parameter distributions defined on the positive half-line.

The resulting exponentiated exponential approximation is

$$f_{\text{Approx},ee,t}(s) = \alpha\theta\left[1-\exp(-\theta s)\right]^{\alpha-1}\exp(-\theta s)\sum_{r=0}^{t} c_r s^r, \tag{4}$$

where c_r are given by (2) with

$$m_i = \frac{(-1)^i \alpha\Gamma(\alpha)}{\theta^i}\frac{\partial^i}{\partial p^i}\left[\frac{\Gamma(p+1-\alpha)}{\Gamma(p+1)}\right]\Big|_{p=\alpha}$$

and $\mu(a)$ are given by (3). In particular,

$$m_1 = \theta^{-1}\left[\psi(\alpha+1)-\psi(1)\right],$$
$$m_2 = \theta^{-2}\left\{\psi'(1)-\psi'(\alpha+1)+\left[\psi(\alpha+1)-\psi(1)\right]^2\right\},$$

where $\psi(x) = d\log\Gamma(x)/dx$ and $\psi'(x) = d\psi(x)/dx$. Solving $m_1=\mu(1)$ and $m_2=\mu(2)$, we obtain α as the root of

$$\frac{\sqrt{\psi'(1)-\psi'(x+1)}}{\psi(x+1)-\psi(1)} = \frac{\sqrt{\mu(2)-\mu^2(1)}}{\mu(1)},$$

and $\theta = \left[\psi(\alpha+1)-\psi(1)\right]/\mu(1)$.

The resulting Weibull approximation is

$$f_{\text{Approx},Weibull,t}(s) = \alpha\theta^{-\alpha}s^{\alpha-1}\exp\left[-(s/\theta)^{\alpha}\right]\sum_{r=0}^{t} c_r s^r, \tag{5}$$