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Exact three-point difference scheme for singular nonlinear boundary value problems

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ABSTRACT

Exact three-point difference scheme on an irregular grid for numerically solving boundary value problems for second order nonlinear ordinary differential equations with a singularity of the first kind is constructed. We have proved the existence and uniqueness of the solution of this scheme as well as have shown the convergence of the associated iteration method. In order to determine the coefficients and the right-hand side of the exact difference scheme at an arbitrary node of the grid, some auxiliary initial value problems on a small interval around this node must be solved. If these initial value problems are solved numerically by a one-step method of high-order accuracy, truncated three-point difference schemes of high-order accuracy results. The effectiveness of proposed difference schemes is illustrated by a numerical example.

1. Introduction

Difference schemes for solving boundary value problems (BVPs) for ordinary differential equations with a singularity of the first kind are considered in [1–4]. However, in these papers difference schemes have a low order of accuracy.

In accordance with approach [5–7], the exact three-point difference scheme (ETDS) for nonlinear BVPs

$$\frac{1}{x}\frac{d}{dx}\left[xk(x)\frac{du}{dx}\right] = -f(x,u), \quad x \in [0,R], \qquad \lim_{x\to 0} xk(x)\frac{du}{dx} = 0, \quad u(R) = \mu_2$$

has been constructed in [8]. Moreover, on the basis of the ETDS an algorithm for the construction of three-point difference schemes (TDS) of high order accuracy is developed in [9].

In the present paper, we constructed the ETDS for the following singular BVPs

$$\begin{aligned} &\frac{1}{x^2}\frac{d}{dx}\left[x^2k(x)\frac{du}{dx}\right] = -f(x,u), \quad x \in [0,R], \\ &\lim_{x\to 0} x^2k(x)\frac{du}{dx} = 0, \quad u(R) = \mu_2 \end{aligned} \tag{1}$$

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on the irregular grid

$$\widehat{\omega}_h = \{x_j \in [0, R], j = 0, 1, \dots, N, x_0 = 0, x_N = R\}.$$

The practical realization of the ETDS for all $x_j, j = 1, 2, \dots, N-1$ can be achieved by the integration of the four auxiliary initial value problems (IVPs): two nonlinear ordinary differential equations and two linear ordinary differential equations on the intervals $[x_{j-1}, x_j]$ (forward) and $[x_j, x_{j+1}]$ (backward). These IVPs can be solved by executing only one step with an arbitrary one-step method (e.g., the Taylor series method or a Runge–Kutta method). Instead of the EDTS we now have truncated TDS. The accuracy of the truncated TDS is determined by the accuracy of the corresponding IVP-solvers.

2. The existence and uniqueness of the solution

The next theorem gives sufficient conditions for the existence and uniqueness of the solution of BVP (1), that is based on Banach's Fixed Point Theorem (see, e.g., [10]).

Theorem 2.1. *Let the following assumptions be satisfied:*

$$0 < c_1 \le k(x) \le c_2 \quad \forall x \in [0, R],\ k(x) \in \mathbb{Q}^1[0, R], \tag{2}$$

$$f_u(x) \equiv f(x, u) \in \mathbb{Q}^0[0, R],\ |f(x, u)| \le K \quad \forall x \in [0, R],\ u \in \Omega([0, R], r), \tag{3}$$

$$|f(x, u) - f(x, v)| \le L|u - v| \quad \forall x \in [0, R],\ u, v \in \Omega([0, R], r), \tag{4}$$

$$q = \frac{LR^2}{6c_1} \max(1, 2Rc_1) < 1. \tag{5}$$

Then, the BVP (1) has a unique solution $u(x) \in \Omega([0, R], r)$, that can be found by the fixed point iterations

$$\begin{gathered} \frac{1}{x^2}\frac{d}{dx}\left[x^2 k(x)\frac{du^{(n)}}{dx}\right] = -f(x, u^{(n-1)}(x)), \quad x \in (0, R), \\ \lim_{x\to 0} x^2 k(x)\frac{du^{(n)}(x)}{dx} = 0, \qquad u^{(n)}(R) = \mu_2, \quad n = 1, 2, \dots,\ u^{(0)}(x) = \mu_2. \end{gathered} \tag{6}$$

Moreover, the error estimate

$$\left\|u^{(n)} - u\right\|^*_{1,\infty,[0,R]} \le \frac{q^n}{1-q} r \tag{7}$$

holds.

Here $\mathbb{Q}^p[0, R]$ is the class of functions with piecewise continuous derivatives up to order p with a finite number of discontinuity points of first kind and

$$\Omega([0, R], r) = \left\{u(x) : u \in W^1_\infty[0, R],\ u, x^2 k(x)\frac{du}{dx} \in \mathbb{C}[0, R],\ \left\|u - u^{(0)}\right\|^*_{1,\infty,[0,R]} \le r\right\},$$

$$r = \frac{KR^2}{6c_1}\max(1, 2Rc_1), \qquad \|u\|_{0,\infty,[0,R]} = \max_{x\in[0,R]} |u(x)|,$$

$$\|u\|^*_{1,\infty,[0,R]} = \max\left\{\|u\|_{0,\infty,[0,R]},\ \left\|x^2 k(x)\frac{du}{dx}\right\|_{0,\infty,[0,R]}\right\}.$$

3. Existence of an exact three-point difference scheme

On the closed interval $[0, R]$ we introduce the irregular grid

$$\widehat{\omega}_h = \{x_j \in [0, R], \quad j = 0, 1, \dots, N,\ x_0 = 0,\ x_N = R\},$$

$$h_j = x_j - x_{j-1} > 0, \quad j = 1, 2, \dots, N, \qquad \hbar_j = \frac{h_j + h_{j+1}}{2}, \quad |h| = \max_{1\le j\le N} h_j$$

such that the discontinuity points of functions $k(x), f(x, u)$ coincide with the grid nodes. We denote the set of all discontinuity points by ρ and choose N so that $\rho \subseteq \widehat{\omega}_h = \{x_j, j = 1, 2, \dots, N-1\}$. The continuity conditions

$$u(x_i - 0) = u(x_i + 0), \qquad x^2 k(x)\frac{du}{dx}\bigg|_{x=x_i-0} = x^2 k(x)\frac{du}{dx}\bigg|_{x=x_i+0} \quad \forall x_i \in \rho$$

have been satisfied at the discontinuity points.