Journal of Computational and Applied Mathematics 294 (2016) 23–38

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Involutions of polynomially parametrized surfaces

Juan Gerardo Alcázar [*,1], Carlos Hermoso

Departamento de Física y Matemáticas, Universidad de Alcalá, E-28871 Madrid, Spain

ARTICLE INFO

Article history:
Received 18 March 2014
Received in revised form 2 August 2015

Keywords:
Algebraic surfaces
Symmetry detection
Involutions
Polynomial parametrizations
Geometric invariants
Symmetries of algebraic surfaces

ABSTRACT

We provide an algorithm for detecting the involutions leaving a surface defined by a polynomial parametrization invariant. As a consequence, the symmetry axes, symmetry planes and symmetry center of the surface, if any, can be determined directly from the parametrization, without computing or making use of the implicit representation. The algorithm is based on the fact, proven in the paper, that any involution of the surface comes from an involution of the parameter space $\mathbb{R}^2$; therefore, by determining the latter, the former can be found. The algorithm has been implemented in the computer algebra system Maple 18. Evidence of its efficiency for moderate degrees, examples and a complexity analysis are also given.

1. Introduction

Symmetry detection in 3D objects is an important matter in fields like Computer Graphics or Computer Vision. In Computer Graphics, it is useful in order to gain understanding when analyzing pictures, and also to perform tasks like compression, shape editing or shape completion. In Computer Vision, symmetry is important for object detection and recognition. Many techniques have been tried so far to solve the problem. Some of them involve statistical methods and, in particular, clustering; see for example the papers [1–4], where the technique of transformation voting is used, or [5], based on the Extended Gauss Image. Other techniques are robust auto-alignment [6], spherical harmonic analysis [7], feature points [8], primitive fitting [9], and spectral analysis [10], to quote just a few. In addition, there are algorithms for computing the symmetries of 2D and 3D discrete objects [11–14] and for boundary-representation models [14–16]. The list of all papers addressing the subject is really very long, and the interested reader is referred to the bibliographies in these papers to get a more complete list.

In the references on the topic, the object to be analyzed is quite commonly a point cloud or a mesh, sometimes with missing parts, so that little structure is assumed on it. One exception here is the case of tensor product surfaces [11]. In this case the geometry of the object, and in particular its symmetry, follows from that of an underlying discrete object, the *control polyhedron*. Hence, the symmetries of the object can be found by applying methods to detect symmetries of discrete objects [11–14].

In this paper we consider the problem of computing *involutions*, i.e. symmetries with respect to a point, line or plane, of objects with a stronger structure, namely the set S of points defined by a polynomial parametrization

$$\boldsymbol{x}(t,s) = (x(t,s), y(t,s), z(t,s)),$$

* Corresponding author.
E-mail addresses: juange.alcazar@uah.es (J.G. Alcázar), carlos.hermoso@uah.es (C. Hermoso).

[1] Member of the Research Group ASYNACS (Ref. CCEE2011/R34).

http://dx.doi.org/10.1016/j.cam.2015.08.002

with $(t, s) \in \mathbb{R}^2$. Such objects, well-known in Constructive Algebraic Geometry and Computer Aided Geometric Design, are called *polynomially parametrized* algebraic surfaces. Certainly a tensor product surface corresponds to this description whenever (t, s) is restricted to a compact rectangle $[a, b] \times [c, d] \subset \mathbb{R}^2$. However, in our case (t, s) takes values over the whole plane $\mathbb{R}^2$. Therefore we deal with the global surface S, not just with a piece of it, and an approach like [11–14] is not applicable here.

In order to solve the problem we assume *good* properties on the parametrization $\boldsymbol{x}(t, s)$. More precisely, we assume that the parametrization is injective except perhaps for a closed subset of (possibly singular) points of S, and that it is also surjective as a mapping from the plane to S. Under these conditions, we prove that any involution of the surface is the result of lifting an involution of the plane to S via the parametrization of the surface. This way, the problem is translated to the parameter space, and in turn reduces to solving bivariate real polynomial systems.

The method can be seen as the generalization to surfaces of some ideas recently applied to compute symmetries of planar and space rational curves [17–20]. Furthermore, the problem treated here is related to the more general question of extracting geometric invariants from a surface parametrization. This question appears as one of the eight open problems on the interplay between Algebraic Geometry and Geometric Modeling posed by Prof. Ron Goldman in [21].

This paper has the following structure. In Section 2 we introduce some generalities on surface parametrizations and isometries, and we prove several results on symmetries of surfaces; although rotational symmetry is not addressed in this paper, some properties of this type of symmetry are considered here, and then used to prove certain facts on involutions. The method itself is presented in Section 3. Section 4 briefly addresses the special case of cylindrical surfaces. Finally, in Section 5 we provide two detailed examples, we address complexity issues, and we report on the practical implementation of the algorithm carried out in the computer algebra system Maple 18. Our conclusions, and some observations about future work, are provided in Section 6.

2. Generalities

2.1. Properness and normality

Throughout this paper we consider an algebraic surface $S \subset \mathbb{R}^3$ different from a plane, polynomially parametrized by $\boldsymbol{x} : \mathbb{R}^2 \to \mathbb{R}^3$, where

$$\boldsymbol{x}(t, s) = (x(t, s), y(t, s), z(t, s))$$

and $x(t, s), y(t, s), z(t, s)$ are polynomials in the variables t, s with coefficients in $\mathbb{Q}$. Nevertheless, at certain points of this paper we will implicitly assume that the parametrization $\boldsymbol{x}$ can also be considered as $\boldsymbol{x} : \mathbb{C}^2 \to \mathbb{C}^3$, so that both the parameter space and the surface can be embedded into the complex plane and the complex space. We will also assume the same thing for other real mappings in the paper. Since S admits a rational, and in fact a polynomial, parametrization then in particular S is irreducible. The functions $x(t, s), y(t, s), z(t, s)$ are the *components* of $\boldsymbol{x}$, while t, s are the *parameters* of $\boldsymbol{x}$. We define the *total degree*, n, of the parametrization $\boldsymbol{x}$ as the maximum of the total degrees of the components of $\boldsymbol{x}$. Furthermore, we will assume that $\boldsymbol{x}$ is *proper*, i.e. birational or equivalently injective for almost all points of S except for at most a closed subset of S. In particular, this implies that $\boldsymbol{x}^{-1}$ is a rational map. One can check properness by using the algorithms in [22,23]; for reparametrization questions one can see [24–27].

We say that the parametrization $\boldsymbol{x}(t, s)$ is *normal* if it is surjective, i.e. if every point of S is reached via $\boldsymbol{x}$ by some pair of parameters $(t, s) \in \mathbb{C}^2$. This problem has been well studied for the case of rational curves [28]. However, the same problem for surfaces is not yet completely well understood. The question has been addressed in [29,30] for special kinds of surfaces, and also in [31], where partial results on the problem are presented. In particular, in [31] a sufficient condition for a polynomial parametrization to be normal (see Corollary 3.15 and Corollary 4.4 therein) is given. Throughout this paper, we will also assume that the parametrization $\boldsymbol{x}$ we work with is normal.

Additionally, for technical reasons we will require $\boldsymbol{x}(0)$ to be a regular point of S; this requirement can always be satisfied by applying, if necessary, a random linear change of parameters.

2.2. Isometries of algebraic surfaces

2.2.1. Basic definitions

Let us recall some facts from Euclidean geometry [32]. An *isometry* of $\mathbb{R}^n$ is a map $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ preserving Euclidean distances. Any isometry f of $\mathbb{R}^n$ is linear affine, taking the form

$$f(x) = Qx + \boldsymbol{b}, \quad x \in \mathbb{R}^n, \tag{1}$$

with $\boldsymbol{b} \in \mathbb{R}^n$ and $Q \in \mathbb{R}^{n\times n}$ an orthogonal matrix. In particular $\det(Q) = \pm 1$. For $n = 3$, the isometries of the space form a group under composition that is generated by reflections, i.e., symmetries with respect to a plane, also called *mirror symmetries*. An isometry is called *direct* when it preserves the orientation, and *opposite* when it does not. In the former case $\det(Q) = 1$, while in the latter case $\det(Q) = -1$. The identity map $\mathrm{id}_{\mathbb{R}^n}$ of $\mathbb{R}^n$ is called the *trivial symmetry*. An isometry $f(x) = Qx + \boldsymbol{b}$ of $\mathbb{R}^n$ is called an *involution* if $f \circ f = \mathrm{id}_{\mathbb{R}^n}$, in which case $Q^2 = I$ is the identity matrix and $\boldsymbol{b} \in \ker(Q + I)$.