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Error bounds for GMLS derivatives approximations of Sobolev functions

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ABSTRACT

This paper provides the error estimates for generalized moving least squares (GMLS) derivatives approximations of a Sobolev function in L^p norms and extends them for local weak forms of DMLPG methods. Sometimes they are called *diffuse* or *uncertain* derivatives, but precisely they are *direct* approximants of exact derivatives which possess the optimal rates of convergence. GMLS derivatives approximations are different from the *standard* derivatives of MLS approximation. While they are much easier to evaluate at considerably lower cost, in this paper the same orders of convergence with comparison to the standard derivatives are obtained for them.

1. Introduction

The moving least squares approximation (MLS) in the current form was introduced by Lancaster and Salkauskas [1] in 1981. There are many researches concerning the error analysis of this approximation. For example see [2–12]. In recent years many authors have tried to improve and develop the MLS approximation in different aspects. As such improvements and developments we can mention the complex variable MLS approximation [13] and the interpolating MLS [14,15].

A presentation of generalized moving least squares (GMLS) approximation and a connection to Backus–Gilbert optimality were done in [4] and then an application to numerical integration was performed in [16]. In [17], the concept of GMLS was linked to the so-called *diffuse derivatives* [18,19] and an error bound in L^∞ norm was derived. The authors of [17] suggested to ignore the phrase "diffuse derivatives" in favor of "GMLS derivatives approximations" because there is nothing diffuse or uncertain about them. Afterward, in [20,21] the concept of GMLS approximation was employed to accelerate the *meshless local Petrov–Galerkin (MLPG) methods* of Atluri and his collaborators [22,23]. The new methods were called *direct MLPG (DMLPG)* because GMLS directly approximates the local weak forms and boundary operators without any detour via classical MLS shape functions.

The optimal rate of convergence for GMLS derivatives approximations toward the exact derivatives has been proved in [17] in $L^\infty(\Omega)$ for sufficiently smooth functions over Ω^*, where Ω^* can be larger than the consideration domain Ω. In this paper we estimate the errors in $L^p(\Omega)$, $p \in [1, \infty]$, relax the bounds for Sobolev functions over Ω and then extend them for local weak forms of DMLPG. The results of this paper can be used for analyzing DMLPG and all methods based on diffuse derivatives.

When GMLS is applied to recover the value of a functional, it suffices to evaluate the functional on a space of polynomials, not on a certain trial space spanned by MLS shape functions. This significantly speeds up numerical calculations, if the functional is complicated, e.g. a high order derivative or a numerical integration against a test function. This is the main advantage of GMLS approximation compared with the MLS approximation. For more details see [17,20,24].

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The rest of this paper is organized as follows. In Section 2 the concept of GMLS approximation is reviewed and in Section 3 error bounds in L^p are proved. In Section 4 the error estimations and convergence rates are justified by some numerical experiments.

2. GMLS derivative approximation

Let $\Omega \subset \mathbb{R}^d$, for positive integer d, be a nonempty and bounded set. In next section, more conditions on Ω will be considered. Assume,

$$X = \{x_1, x_2, \ldots, x_N\} \subset \Omega,$$

is a set containing N scattered points, called *centers* or *data site*. Distribution of points should be well enough to pave the way for analysis.

Henceforth, we use $\mathbb{P}_m^d$, for $m \in \mathbb{N}_0 = \{n \in \mathbb{Z}, n \geqslant 0\}$, as space of d-variable polynomials of degree at most m of dimension $Q = \binom{m+d}{d}$. A basis for this space is denoted by $\{p_1, \ldots, p_Q\}$.

The MLS, as a meshless approximation method, provides an approximation $\widehat{u}(x)$ of $u(x)$ in terms of values $u(x_j)$ at centers x_j by

$$u(x) \approx \widehat{u}(x) = \sum_{j=1}^{N} a_j(x) u(x_j), \quad x \in \Omega,$$

where a_j are *MLS shape functions*. MLS finds the best approximation to u out of $\mathbb{P}_m^d$ with respect to a discrete ℓ^2 norm induced by a *moving* inner product, where the corresponding weight function w depends not only on points x_j but also on point x to be approximated. Indeed, the influence of the centers is governed by $w(x, x_j)$, which vanishes for arguments $x, x_j \in \Omega$ with $\|x - x_j\|_2$ greater than a certain threshold, say δ. Thus we can define $w(x, x_j) = \varphi(\|x - x_j\|_2/\delta)$ where $\varphi : \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ is a compactly supported function on $[0, 1]$.

Derivatives of u are usually approximated by derivatives of $\widehat{u}$,

$$D^\alpha u \approx D^\alpha \widehat{u}(x) = \sum_{j=1}^{N} D^\alpha a_j(x) u(x_j), \quad x \in \Omega,$$

and they are called *standard derivatives*. Since derivatives of complicated and non-closed from shape functions a_j should be taken, the standard derivatives are known to be time-consuming. This is the reason why some people avoid using them and take a bypass via diffuse derivatives [18,19].

Another approach is a direct approximation of $D^\alpha u$ from the data without detour via derivatives of $\widehat{u}$. In this case we have

$$D^\alpha u \approx \widehat{D^\alpha u}(x) = \sum_{j=1}^{N} a_{j,\alpha}(x) u(x_j), \quad x \in \Omega. \tag{2.1}$$

This is a GMLS approximation where $D^\alpha u$ is recovered directly from $u(x_j)$ as a linear functional. It should be noted that $D^\alpha a_j(x) \neq a_{j,\alpha}(x)$ in general, and in fact in vector form

$$\boldsymbol{a}_\alpha(x) = WP^T(PWP^T)^{-1} D^\alpha \boldsymbol{p},$$

where W is the diagonal matrix carrying the weights $w_j = w(x, x_j)$ on its diagonal, P is $N \times Q$ matrix of values $p_k(x_j)$, $1 \leqslant j \leqslant N$, $1 \leqslant k \leqslant Q$ and $\boldsymbol{p} = (p_1, \ldots, p_Q)^T$. It is clear that the operator D^α acts only on the basis polynomials $\boldsymbol{p}$, and this significantly reduces the cost of computations. Details are provided in [17]. This approach provides the GMLS derivatives approximations and [17] shows the coincidence with diffuse derivatives and gives an error bound in L^∞ for them.

Eq. (2.1) can even be extended to more general recovery problem: under some conditions on linear functional λ, we can write

$$(\lambda u)(x) \approx \widehat{\lambda u}(x) = \sum_{j=1}^{N} a_{j,\lambda}(x) u(x_j), \tag{2.2}$$

where the functional can be for instance point evaluations, derivative or integral operators, etc. Here $a_{j,\lambda}$ are functions associated with λ and in vector form they can be obtained by

$$\boldsymbol{a}_\lambda = WP^T(PWP^T)^{-1} \lambda(\boldsymbol{p}).$$

Thus it suffices to evaluate λ on the space $\mathbb{P}_m^d$, not on a certain trial space spanned by certain shape functions. This significantly speeds up numerical calculations, if the functional λ is complicated, e.g. a numerical integration against a test function. This generalized approximation is the building block of different variations of DMLPG method [20,21,24].