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Iterative refinement for a system of linear integro-differential equations of fractional type

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ABSTRACT

A novel technique based on iterative refinement is developed to approximate the analytical solution of a system of linear fractional integro-differential equations. While the study focuses mainly on Fredholm-type equations, adaptation to the Volterra-type is also presented. A comparison is made with the method of successive approximations on the basis of convergence speed and accuracy. Several numerical examples are given to demonstrate the efficacy of our algorithm. The authors also present formulations of some error bounds.

1. Introduction

Fractional differential and integral equations are nowadays the subject of intensive research due to their frequent appearance in many engineering and scientific disciplines. Although the notion of fractional calculus dates back three centuries ago, it started to flourish only in the last three decades when engineers and scientists developed models which involve fractional operators. Several techniques were adapted to obtain approximate analytical solutions for fractional differential and integral equations, and needless to say integro-differential equations. Recently, methods like the Laplace Transform and the Mellin Transform – which were one day classical in nature – began to flourish very rapidly. One of the most popular methods which attracted the attention of many mathematicians is the Adomian decomposition method [1,2]. Among the other methods are the differential transform method [3], Taylor expansion methods [4,5], spectral methods [6], as well as the variational iteration and homotopy perturbation methods [7,8] – only to name a few. The widespread of these methods enabled the subject of fractional calculus to become a well established discipline in applied mathematics, and thus finding its way into the toolbox of many commercial packages like MATLAB and MATHEMATICA.

The method of *iterative refinement* lies within the scope of the above analytical methods. The concept itself is simple and has long been known to mathematicians as well as engineers. The main idea is that one uses an approximate solution – which is updated based upon a computed residual – to adjust the system input. The algorithm is repeated until certain accuracy is reached. In mathematics, a set of linear algebraic equations $Ax = b$ whose solution is computed as $\bar{x}$ (due to rounding) can be corrected by forming the residual error $r = A\bar{x} - b$. The approximate solution $\bar{x}$ is then updated by subtracting a quantity δx from $\bar{x}$ obtained from solving $A\delta x = r$. So it is mainly a residual correction approach as some authors prefer to call it [9,10]. In engineering, the concept is similar and is represented by negative feedback mechanisms. Typical examples exist, for instance, in refrigeration; in which a negative feedback controller uses an error signal to adjust the system's temperature. The error signal is given by the difference between the actual and the desired temperature.

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In this paper, the concept of iterative refinement is applied for the first time to a system of linear fractional integro-differential equations of the 2nd type. A comparison is made with the method of successive approximations on the basis of convergence speed and accuracy. For the sake of brevity, IR will denote the method of iterative refinement against SA for the successive approximations.

The system under study will be of the form

$$\begin{aligned}
D^{\alpha_1}\psi_1(x) &= f_1(x) + \lambda_1\left(\int_a^b k_{11}(x,s)\psi_1(s)\,ds + \cdots + \int_a^b k_{1n}(x,s)\psi_n(s)ds\right),\\
D^{\alpha_2}\psi_2(x) &= f_2(x) + \lambda_2\left(\int_a^b k_{21}(x,s)\psi_1(s)\,ds + \cdots + \int_a^b k_{2n}(x,s)\psi_n(s)\,ds\right),\\
&\vdots\\
D^{\alpha_n}\psi_n(x) &= f_n(x) + \lambda_n\left(\int_a^b k_{n1}(x,s)\psi_1(s)\,ds + \cdots + \int_a^b k_{nn}(x,s)\psi_n(s)\,ds\right),
\end{aligned} \tag{1}$$

where D^{α_i}, which acts as ${}_aD_x^{\alpha_i}$, is the fractional differential operator w.r.t. x of order $\alpha_i > 0,\ i = 1, 2, \ldots, n$ in the Caputo sense, and subject to the initial conditions $\psi_i^{(j)}(a),\ j = 0, 1, \ldots, m_i - 1$ with $m_i = \lceil\alpha_i\rceil$, the latter being the smallest integer not smaller than α_i. The integral operators in (1) define a set of completely continuous linear operators in the space of continuous functions $C[a, b]$ equipped with the sup norm.

2. A Fredholm integro-differential equation with fractional derivative

Since generalization to n equations is straight-forward, we start by considering one equation. Hence, for the equation

$$D^{\alpha}\psi(x) = f(x) + \lambda\int_a^b k(x,s)\psi(s)\,ds \tag{2}$$

with initial conditions $\psi^{(j)}(a),\ j = 0, 1, \ldots, m-1$ with $m = \lceil\alpha\rceil$, we get the following by fractional integration [11]

$$\begin{aligned}
\psi(x) &= \sum_{j=0}^{m-1}\frac{(x-a)^j\psi^{(j)}(a)}{j!} + \frac{1}{\Gamma(\alpha)}\int_a^x\frac{f(t)}{(x-t)^{1-\alpha}}dt + \frac{\lambda}{\Gamma(\alpha)}\int_a^x\int_a^b\frac{k(t,s)}{(x-t)^{1-\alpha}}\psi(s)\,ds dt\\
&= \sum_{j=0}^{m-1}\frac{(x-a)^j\psi^{j}(a)}{j!} + \frac{1}{\Gamma(\alpha)}\int_a^x\frac{f(t)}{(x-t)^{1-\alpha}}dt + \frac{\lambda}{\Gamma(\alpha)}\int_a^b\left(\int_a^x\frac{k(t,s)}{(x-t)^{1-\alpha}}dt\right)\psi(s)\,ds.
\end{aligned} \tag{3}$$

By setting

$$\sum_{j=0}^{m-1}\frac{(x-a)^j\psi^{(j)}(a)}{j!} + \frac{1}{\Gamma(\alpha)}\int_a^x\frac{f(t)}{(x-t)^{1-\alpha}}dt \equiv \tilde{f}(x) \quad\text{and}\quad \frac{1}{\Gamma(\alpha)}\int_a^x\frac{k(t,s)}{(x-t)^{1-\alpha}}dt \equiv \tilde{k}(x,s). \tag{4}$$

Eq. (3) can be written in the abbreviated Fredholm integral equation of the 2nd type

$$\psi = \tilde{f} + \lambda\tilde{K}\psi, \quad \text{i.e. } (I - \lambda\tilde{K})\psi = \tilde{f} \quad\text{with}\quad \tilde{K}\psi = \int_a^b\tilde{k}(\cdot,s)\psi(s)\,ds, \tag{5}$$

and a kernel $\tilde{k}$ bounded on the square $[a, b]\times[a, b]$. The functions $\tilde{f}$ and ψ are in $C[a, b]$, equipped with the sup norm which we denote as $\|.\|$. Consequently, the operator norm for $\tilde{K}$ is (e.g. [12, p. 11])

$$\|\tilde{K}\| = \max_{x\in[a,b]}\int_a^b|\tilde{k}(x,s)|ds. \tag{6}$$

One possible method for solving (5) is to iterate using the successive approximations

$$\psi_n = \tilde{f} + \lambda\tilde{K}\psi_{n-1} \tag{7}$$

which converges to ψ if $|\lambda|\,\|\tilde{K}\| < 1$.