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Heisenberg uncertainty principle for a fractional power of the Dunkl transform on the real line

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ABSTRACT

The aim of this paper is to prove Heisenberg–Pauli–Weyl inequality for a fractional power of the Dunkl transform on the real line for which there is an index law and a Plancherel theorem.

1. Introduction

Dunkl operators are differential–difference operators associated with finite reflection groups in a euclidean space. The first class of such operators was introduced by C.F. Dunkl in a series of papers [1–3], where he built up the framework for a theory of special functions and integral transforms in several variables related with reflection groups. In addition to the multidimensional case, one-dimensional Dunkl operators are also of great interest. For example, a number of works have recently appeared that develop the harmonic analysis results associated with the one-dimensional Dunkl operator. One of them is the Heisenberg–Weyl type inequality for the one-dimensional Dunkl transform established by Rösler and Voit [4].

The objective of this paper is two-folded: firstly, we develop a harmonic analysis related to a Dunkl type operator on the real line. More precisely, we consider a singular differential–difference operator Λ_μ^α on $\mathbb{R}$ which includes as a particular case the one-dimensional Dunkl operator. The eigenfunction $K_{\mu,\alpha}$ of this operator permits to define a fractional power D_μ^α of the Dunkl transform on $\mathbb{R}$ that reduces to the Dunkl transform, fractional Hankel transform and the fractional Fourier transform for particular cases of the parameters. Next, we develop an L^1 and L^2 theory for this transform. For L^1 theory, we give Riemann–Lebesgue lemma, inversion formula, index additivity property, which is of central importance: without it, we could hardly think of D_μ^α as being the αth power of D_μ, and operational formula. As for as L^2 theory, we prove that the fractional Dunkl transform D_μ^α, initially defined on $L^1(\mathbb{R}, |x|^{2\mu+1}dx)$, has a unique extension to a unitary operator of $L^2(\mathbb{R}, |x|^{2\mu+1}dx)$ and if the extension is also denoted by D_μ^α then the family $\left\{D_\mu^\alpha\right\}_{\alpha\in\mathbb{R}}$ which is parameterized by the parameter $\alpha \in \mathbb{R}$ has a group structure, called the elliptic group. It is like a rotation group since $D_\mu^\alpha \circ D_\mu^\beta = D_\mu^{\alpha+\beta}$ and D_μ^0 is the identity and the inverse is obviously $\left(D_\mu^\alpha\right)^{-1} = D_\mu^{-\alpha}$. We present also the subject of eigenvalues and eigenfunctions. We show that the generalized

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Hermite functions, which were introduced by Szegö [5] and studied by Chihara [6,7] and Rosenblum [8], form an orthonormal basis of eigenfunctions of D_μ^α on $L^2(\mathbb{R}, |x|^{2\mu+1}dx)$. As a consequence, we prove that the family $\{D_\mu^\alpha\}_{\alpha\in\mathbb{R}}$ is a $\mathcal{C}_0$-group of unitary operators and we derive their infinitesimal generators. Secondly, we extend the Heisenberg–Pauli–Weyl uncertainty inequalities established by Rösler and Voit [4, Theorem 4.1] to the case of fractional Dunkl transform D_μ^α as follows:

$$\mathrm{var}_\mu(D_\mu^\alpha(f))\,\mathrm{var}_\mu(D_\mu^\beta(f)) \geq \sin^2(\alpha-\beta)\left(\left(\mu+\frac{1}{2}\right)\left(\|f_e\|_{2,\mu}^2-\|f_o\|_{2,\mu}^2\right)+\frac{1}{2}\right)^2, \tag{1.1}$$

where $\alpha \in \mathbb{R}\backslash\pi\mathbb{Z}$ and $\beta \in \mathbb{R}$. For this purpose, we introduce Sobolev type spaces $H_2^{\mu,\alpha}(\mathbb{R})$ naturally associated to $\Lambda_\mu^{-\alpha}$ and we obtain their basic properties such as the embedding theorems. We prove that:

- For every $\alpha \in \mathbb{R}\backslash\pi\mathbb{Z}$ and $-\frac{1}{2} \leq \mu < 0$, $H_2^{\mu,\alpha}(\mathbb{R}) \hookrightarrow \mathcal{C}_0(\mathbb{R})$ and the injection map is continuous.
- For every $\alpha \in \mathbb{R}\backslash\pi\mathbb{Z}$, $\mu \geq 0$ and $f \in H_2^{\mu,\alpha}(\mathbb{R})$, there exists a function $\psi \in \mathcal{C}(\mathbb{R}\backslash\{0\})$ such that $f(x) = \psi(x)$, a.e. and for all $x \in \mathbb{R}\backslash\{0\}$,

$$|\psi(x)| \leq c\,\|f|H_2^{\mu,\alpha}(\mathbb{R})\|_{2,\mu}\begin{cases}|x|^{-\mu} & \text{if } \mu>0,\\ |\ln|x|\,|^{\frac{1}{2}} & \text{if } \mu=0,\end{cases}$$

  where $c = c(\mu,\alpha) > 0$. As applications on these spaces, we will show that (1.1) holds with equality if and only if $f(x) = \lambda\, e^{(i\cot(\beta)-b)\frac{x^2}{2}}E_\mu(ax)$, where λ, a and b are suitable parameters.

This paper is organized as follows. Section 2 presents an overview of the Heisenberg's inequality for various Fourier transform on the real line. Section 3 we introduce the fractional Dunkl transform D_μ^α on the real line with parameter $\alpha \in \mathbb{R}$. Riemann–Lebesgue lemma, inversion formula, an index additivity property and operational formulae are derived in Section 4. Section 5 is devoted to the extension of the fractional Dunkl transform D_μ^α as an isometry from $L_\mu^2(\mathbb{R})$ to itself and the intimate relationship between the fractional Dunkl transform and generalized Hermite polynomials and functions. In Section 6, we study the Sobolev spaces $H_2^{\mu,\alpha}(\mathbb{R})$ associated to $\Lambda_\mu^{-\alpha}$ and we derive a Heisenberg uncertainty principle for the fractional Dunkl transform and the fractional Hankel transform.

2. A brief survey of the Heisenberg's inequality for various Fourier transform

In this section, we give an overview of the Heisenberg's inequality for various Fourier transform on the real line.

2.1. The Heisenberg's inequality for Fourier transform and fractional Fourier transform

• The Fourier transform (FT) can be defined in many ways. For us, three different formulations are in particular important. In its most formulation, the FT is given by the integral transform

$$\mathcal{F}(f)(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)e^{-ix\xi}\,dx.$$

Alternatively, one can rewrite the transform as

$$\mathcal{F}(f)(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} K(x,\xi)f(x)\,dx, \tag{2.1}$$

where $K(x,\xi)$ is the unique solution of the system of PDEs

$$\begin{cases}\dfrac{d}{dx}K(x,\xi) = -i\xi\,K(x,\xi),\\ k(0,\xi)=1.\end{cases} \tag{2.2}$$

A third formulation is given by

$$\mathcal{F} = e^{\frac{i\pi}{4}}e^{\frac{i\pi}{4}(\Delta-x^2)}, \tag{2.3}$$

with $\Delta = \frac{d^2}{dx^2}$. The classical Heisenberg–Pauli–Weyl inequality [9] states that for $f \in L^2(\mathbb{R})$ and for any $a, b \in \mathbb{R}$,

$$\int_{-\infty}^{+\infty}(x-a)^2|f(x)|^2\,dx.\ \int_{-\infty}^{+\infty}(\xi-b)^2|\mathcal{F}(f)(\xi)|^2\,d\xi \geq \frac{1}{4}\left(\int_{-\infty}^{+\infty}|f(x)|^2\,dx\right)^2.$$

It is well known that there is a probabilistic interpretation to the previous inequality in terms of the variance. Let $f \in L^2(\mathbb{R})$ and suppose $\|f\|_{L^2(\mathbb{R})} = 1$. By the Parseval identity, $\|\mathcal{F}(f)\|_{L^2(\mathbb{R})} = 1$. Then $|f|^2$ and $|\mathcal{F}(f)|^2$ are both probability density functions on $\mathbb{R}$. The variance of f and the variance of $\mathcal{F}(f)$ are defined by

$$\mathrm{var}(f) = \inf_{a\in\mathbb{R}}\int_{-\infty}^{+\infty}(x-a)^2|f(x)|^2\,dx, \qquad \mathrm{var}(\mathcal{F}(f)) = \inf_{b\in\mathbb{R}}\int_{-\infty}^{+\infty}(x-b)^2|\mathcal{F}(f)(x)|^2\,dx.$$