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# An efficient algorithm based on eigenfunction expansions for some optimal timing problems in finance

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ABSTRACT

This paper considers the optimal switching problem and the optimal multiple stopping problem for one-dimensional Markov processes in a finite horizon discrete time framework. We develop a dynamic programming procedure to solve these problems and provide easy-to-verify conditions to characterize connectedness of switching and exercise regions. When the transition or Feynman–Kac semigroup of the Markov process has discrete spectrum, we develop an efficient algorithm based on eigenfunction expansions that explicitly solves the dynamic programming problem. We also prove that the algorithm converges exponentially in the series truncation level. Our method is applicable to a rich family of Markov processes which are widely used in financial applications, including many diffusions as well as jump–diffusions and pure jump processes that are constructed from diffusion through time change. In particular, many of these processes are often used to model mean-reversion. We illustrate the versatility of our method by considering three applications: valuation of combination shipping carriers, interest-rate chooser flexible caps and commodity swing options. Numerical examples show that our method is highly efficient and has significant computational advantages over standard numerical PDE methods that are typically used to solve such problems.



## 1. Introduction

Problems in finance often involve timing decisions, and many of them can be formulated as the optimal switching problem or the optimal multiple stopping problem. In the former problem, there are several regimes and the decision maker decides when and where to switch to maximize expected payoffs from each regime, minus any costs incurred for switching. In the latter problem, the decision maker holds multiple exercise rights and her goal is to maximize the expected payoffs from all exercises. This problem is an extension of the classical optimal stopping problem with only one exercise right (see e.g., [1] for the classical optimal stopping problem with applications). Applications of the optimal switching problem include but are not limited to, Brennan and Schwartz [2] and Dixit and Pindyck [3] for the valuation of natural resource mines, Dixit [4] for the production decision of a company, and Sødal et al. [5] for the valuation of combination shipping carriers which can carry different types of cargo. For the multiple stopping problem, two well-known examples are interest-rate chooser flexible caps and floors, and commodity swing options. These derivatives are important tools for managing interest rate risk (e.g., [6,7]) and commodity volume risk (e.g., [8,9]), respectively.

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In this paper, we assume the underlying uncertainty is modeled by a one-dimensional Markov process, and consider finite horizon optimal switching and optimal multiple stopping problems with decisions made in discrete time. This setting is appropriate in many real-world applications. For example, in reality, mining rights expire in finite time and shipping carries have finite useful life. Exercise in discrete time is often a contractual requirement, as in the case of interest-rate chooser flexible caps/floors and some commodity swing options. In the case of combination shipping carriers, it is impossible to realize decisions in continuous time since the ship cannot be switched to carry another type of cargo until it finishes its current trip.

We treat both problems in a unified way as the optimal multiple stopping problem can be formulated as an optimal switching problem with constraints. Under some minimal integrability conditions, we derive a dynamic programming procedure to solve these problems and characterize the optimal strategy.

In general, the dynamic programming problem must be solved numerically. When the underlying uncertainty is modeled by a one-dimensional Markov process, a popular choice in practice is the lattice method due to its intuitiveness and flexibility in incorporating dynamic programming. For example, binomial or trinomial trees are used by Pedersen and Sidenius [6] and Ito et al. [10] for pricing chooser flexible caps and by Thompson [11], Lari-Lavassani et al. [12] and Jaillet et al. [8] for valuation of swing options. More generally, implicit schemes for PDE/PIDE can be used, which are more efficient than the lattice method that corresponds to explicit finite difference schemes.

While numerical PDE/PIDE schemes are general-purpose algorithms, many stochastic models in finance have special features, based on which more efficient computational methods can be developed. An important case is when the characteristic function of the underlying Markov process is known, which is true for Lévy processes in particular. In this case, the method of Fourier-cosine series expansions and fast Hilbert transform are both highly efficient (see [13] for another efficient method). For the development and applications of the Fourier-cosine expansion method, see [14] for European options, Fang and Oosterlee [15] for Bermudan and discretely monitored barrier options, and Zhang and Oosterlee [16] for swing options. The fast Hilbert transform method has been developed and applied by Feng and Linetsky [17,18] for discretely monitored barrier and lookback options, and by Feng and Lin [19] for Bermudan options.

This paper considers another important case in finance where the transition semigroup or Feynman–Kac semigroup of the underlying Markov process defined on the Hilbert space of square-integrable payoffs can be represented by an eigenfunction expansion (see Assumption 1; we consider Feynman–Kac semigroup to accommodate interest rate applications where the short rate is stochastic). Many diffusion processes that are commonly used in financial modeling possess discrete spectrum with explicit eigenvalues and eigenfunctions. Well-known examples include the CEV process [20], the Ornstein–Uhlenbeck (OU) process [21], the CIR process [22], the 3/2 process [23] and the Jacobi process [24]. The last four processes are frequently used to model mean-reversion, which is a key feature in the dynamics of many quantities of interests, such as the short rate, commodity spot prices, exchange rates in a target zone and the price difference between two assets. Moreover, this setting includes a rich class of jump–diffusions and pure jump processes that are constructed from diffusions with discrete spectrum through Bochner's subordination and additive subordination (i.e., time changing diffusions with independent Lévy or additive subordinators). We refer readers to Li and Linetsky [25] and Li et al. [26] for detailed discussions of these processes, which feature state-dependent jumps in general and if additive subordination is used, jumps are also time-dependent. For example, applying Bochner's/additive subordination to a mean-reverting diffusion results in a process with state-dependent jumps which also contribute to mean-reversion. It is shown that these processes improve the realism of diffusion processes while retaining analytical tractability. In particular, models based on Bochner's subordination are able to calibrate volatility smiles of a single maturity while those based on additive subordination can calibrate the entire implied volatility surface. For applications in financial modeling, see [27] for equity, Li and Linetsky [25], Li and Mendoza-Arriaga [28] and Li et al. [26] for commodities, Boyarchenko and Levendorskiĭ [29] and Lim et al. [30] for interest rates, Mendoza-Arriaga et al. [31] for credit-equity derivatives and Mendoza-Arriaga and Linetsky [32] for credit derivatives.

The eigenfunction expansion method is applied by for example, Lewis [33], Davydov and Linetsky [34], Gorovoi and Linetsky [35] and Boyarchenko and Levendorskiĭ [29] for pricing European options, and it is extended by Li and Linetsky [25,36,37] and Lim et al. [30] to solve optimal stopping problems and first passage problems. In many cases, the eigenfunctions are orthogonal polynomials, allowing the method to be efficiently implemented based on the recursion for orthogonal polynomials (see [38] for general discussions on Markov processes and orthogonal polynomials). The present paper applies the eigenfunction expansion method to solve more general and complex optimal timing problems in finance, which are usually solved by numerical PDE/PIDE methods in the literature. We will prove that, under some mild conditions, the eigenfunction expansion algorithm converges exponentially in the series truncation level. To our best knowledge, analysis of the computational property of the eigenfunction expansion method in a dynamic programming setting has not been given in the existing literature. Through numerical examples we will show that our method is highly efficient for finding not only the value function but also the boundary points of the switching/exercise regions, and it has significant computational advantages over numerical PDE/PIDE schemes.

The eigenfunction expansion method is analytical in nature. Assuming the payoff functions and the switching cost functions are square-integrable, we are able to obtain analytical solutions to the value function of the optimal switching problem and the optimal multiple stopping problem through eigenfunction expansions, subject to knowing the switching/exercise regions. The knowledge of these regions is also required in methods based on Fourier-cosine expansions and fast Hilbert transform to solve the dynamic programming problem. In many financial applications, these regions are connected. To find them one just needs to locate the boundary points, which can be done by numerically solving globally