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Occupation times of hyper-exponential jump diffusion processes with application to price step options

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ABSTRACT

We are interested in occupation times of jump diffusion processes with hyper-exponentially distributed jump sizes. We develop a new approach to derive analytical formulas for the Laplace transform of the joint distribution of a hyper-exponential jump diffusion process and its occupation times. These formulas are then used to price step options.

1. Introduction

It has a long history to investigate occupation times of stochastic processes. For a standard Brownian motion $\{W_t; t \geq 0\}$, Lévy [1] derived the following well-known result:

$$\mathbb{P}\left(\int_0^t \mathbf{1}_{\{W_u \geq 0\}} du \in ds\right) = \frac{ds}{\pi\sqrt{s(t-s)}}, \quad 0 < s < t, \tag{1.1}$$

where $\mathbf{1}_A$ is the indicator function of a set A. Then, the corresponding results for a Brownian motion with drift were obtained in [2,3]. After that, for a spectrally negative Lévy process $X = \{X_t; t \geq 0\}$, i.e., a Lévy process without positive jumps, Landriault et al. [4] has derived the Laplace transform of $\int_0^\infty \mathbf{1}_{\{X_t<0\}} dt$. Moreover, for $0 \leq a \leq b \leq c$, the joint Laplace transform of

$$\left(\tau_0^-, \int_0^{\tau_0^-} \mathbf{1}_{\{a<X_t<b\}} dt\right) \quad \text{and} \quad \left(\tau_c^+, \int_0^{\tau_c^+} \mathbf{1}_{\{a<X_t<b\}} dt\right) \tag{1.2}$$

has also been obtained under the assumption that X is a spectrally negative Lévy process (see [5]), where τ_0^- and τ_c^+ are the first passage times of X. Recently, Guèrin and Renaud [6] studied the following expectation:

$$\mathbb{E}_x\left[e^{-p\int_0^t \mathbf{1}_{\{a<X_s<b\}} ds} \mathbf{1}_{\{X_t \in dy\}}\right], \tag{1.3}$$

where X is a spectrally negative Lévy process. They derived formulas for the Laplace transform of (1.3) with respect to t and applied their results to price step options. However, most papers of such research assume that the process X is a diffusion process or a Lévy process without positive jumps.

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In this paper, for a hyper-exponential jump diffusion X, we explore the computation of the following distribution:

$$\mathbb{E}_x\left[e^{-p\int_0^t \mathbf{1}_{\{a<X_s\le b\}}ds}\mathbf{1}_{\{X_t\in dy\}}\right], \tag{1.4}$$

and we succeed in deriving formulas for the Laplace transform of (1.4) with respect to t when $a = -\infty$. Although our approach can also be applied to derive the Laplace transform of (1.4) for $a \in \mathbb{R}$, we do not consider this case in this article because it needs many additional calculations.

We should mention that, under the double exponential jump diffusion process, Cai et al. [7] derived formulas for

$$\int_0^\infty e^{-(a+r)t}\mathbb{E}_x\left[e^{-\rho\int_0^t \mathbf{1}_{\{X_s\le b\}}ds+\gamma X_t}\right]dt, \tag{1.5}$$

with a proper γ. Here, we extend their model to the hyper-exponential jump diffusion process, and more importantly, we emphasize that our approach is completely different from theirs. In [7], they first showed that (1.5), as a function of x, satisfies an integro-differential equation (IDE), then reduced the IDE to an ordinary differential equation (ODE), and finally solved the ODE. However, in this article, our method depends on the strong Markov property of X and the solutions of one-sided and two-sided exit problems of X. Therefore, we do not need to take too much effort as in [7] to establish an IDE. In addition, the assumption of exponential jumps is important in [7] while it is not used in our approach.

For the application of our results, we consider the pricing of occupation time derivatives, specifically, the pricing of step options. Occupation time derivatives are introduced to resolve some questions involved in the standard barrier options. For example, for a knock-out barrier option, its value becomes 0 as soon as the underlying asset price crosses the barrier. This property leads to an obstacle for option trader to manage the risk. For more details about the disadvantages of standard barrier options, see [8]. Unlike the standard barrier option, the payoff of an occupation time derivative often depends on the occupation times of the underlying asset, which helps to alleviate the above problem effectively. There are many types of such derivatives such as step options and corridor options, see for example [7], and many papers study the pricing of them, see among others, [2,8–12].

The rest of the paper is organized as follows. In Section 2, the details of our model and some important preliminary results are given. Our main results are derived in Section 3, and then in Section 4, the application to price step options and some numerical results are presented. Finally, we draw conclusions in Section 5.

2. Model specification and some preliminary outcomes

Consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, P)$ which satisfies the usual hypotheses of completeness and right continuity. The process $X = (X_t)_{t\ge 0}$ is a hyper-exponential jump diffusion process, i.e.,

$$X_t = X_0 + \mu t + \sigma W_t + \sum_{k=1}^{N_t} Y_k, \tag{2.1}$$

where μ and X_0 are constants, $\{W_t;\ t \ge 0\}$ is a standard Brownian motion and $\sigma > 0$ is the volatility of the diffusion; $\{N_t;\ t \ge 0\}$ is a Poisson process with rate λ and $\{Y_k;\ k = 1, 2, \ldots\}$ are independent and identically distributed random variables supported on $\mathbb{R}/\{0\}$; moreover, $\{W_t\}_{t\ge 0}$, $\{N_t\}_{t\ge 0}$ and $\{Y_k;\ k = 1, 2, \ldots\}$ are mutually independent; finally, the probability density function (pdf) of Y_1 is given by

$$f_Y(y) = \sum_{i=1}^m p_i\eta_i e^{-\eta_i y}\mathbf{1}_{\{y>0\}} + \sum_{j=1}^n q_j\vartheta_j e^{\vartheta_j y}\mathbf{1}_{\{y<0\}}, \tag{2.2}$$

where $p_i > 0$, $\eta_i > 0$ for all $i = 1, \ldots, m$, and $\eta_1 < \eta_2 < \cdots < \eta_m$; $q_j > 0$, $\vartheta_j > 0$ for all $j = 1, \ldots, n$, and $\vartheta_1 < \vartheta_2 < \cdots < \vartheta_n$; $\sum_{i=1}^m p_i + \sum_{j=1}^n q_j = 1$. Since X is a Lévy process, we have that X is a strong Markov process (see, e.g., Theorem 3.1 on page 68 in [13]).

In this paper, we denote by $\mathbb{P}_x$ the law of X starting from x and let $\mathbb{E}_x$ represent the corresponding expectation, and when $x = 0$, we write $\mathbb{P}$ and $\mathbb{E}$ for the sake of brevity. In the following, we are interested in occupation times of X and we want to derive formulas for the Laplace transform of (1.4) with respect to t, i.e.,

$$\int_0^\infty qe^{-qt}\mathbb{E}_x\left[e^{-p\int_0^t \mathbf{1}_{\{X_s\le b\}}ds}\mathbf{1}_{\{X_t\in dy\}}\right]dt = \mathbb{E}_x\left[e^{-p\int_0^{e(q)} \mathbf{1}_{\{X_s\le b\}}ds}\mathbf{1}_{\{X_{e(q)}\in dy\}}\right], \tag{2.3}$$

where $p, q > 0$ and $b \in \mathbb{R}$ is a constant; $e(q)$ is an exponential random variable with mean $\frac{1}{q}$ and independent of the process X under $\mathbb{P}_x$. Before starting the derivation in the next section, we first give some useful results in the following.

For the process X defined by (2.1), its Lévy exponent is given by

$$\psi(\theta) := \ln\left(\mathbb{E}\left[e^{\theta X_1}\right]\right) = \frac{\sigma^2}{2}\theta^2 + \mu\theta + \lambda\left(\sum_{i=1}^m \frac{p_i\eta_i}{\eta_i-\theta} + \sum_{j=1}^n \frac{q_j\vartheta_j}{\vartheta_j+\theta} - 1\right), \quad \theta \in (-\vartheta_1, \eta_1). \tag{2.4}$$

The following lemma characterizes the solution of $\psi(\theta) = q$ for $q > 0$ and is taken from Lemma 2.1 in [14].