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Interpolating refinable function vectors with reflection properties

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ABSTRACT

In this paper, we investigate interpolating refinable function vectors whose component functions constitute reflection pairs. A necessary condition for an interpolating refinable function vector to satisfy the reflection property is provided for any dilation factor and multiplicity. Although, the converse is not true in general, for some special cases it becomes a sufficient condition as well. We concentrate on the relation between a dilation factor and multiplicity, and provide the necessary and sufficient condition on the coefficients of its refinement mask in order for such refinable function vector to form a reflection pair. Finally, trivial solutions consisting of translated versions of one component function which is symmetric by itself are discussed. To illustrate the results, various numerical examples are provided.

1. Introduction

Wavelet theory based on refinable functions with interpolating property has been a useful tool in many applications such as sampling theory, signal or image processing, computer graphics, numerical algorithms, and so on. Recently, multiwavelets are used widely for various practical reasons: good performance, smoothness, high approximation order, and so on. In constructing multiwavelet systems, one is faced with solving nonlinear matrix equations. To overcome such difficulties, additional properties are often imposed. Using a two-direction matrix is one possible approach, wherein the filter $\mathbf{P}(\omega)$ consists of two pairs of dual entries.

In this article, we intend to study interpolating refinable function vectors which consist of reflection pairs of their component functions. This is motivated mainly by multiwavelet frames constructed by two-direction refinable functions as in [1–3] and we attempt to redefine this notion by adding useful properties for any dilation factor and multiplicity. One advantage of considering refinable function vectors with the reflection property is that required filter information is significantly reduced in implementation as is the case with those having the symmetry property. Refinable function vectors with the reflection property are of interest in computer-aided geometric design and wavelet-based applications such as digital image processing. The constructed interpolating refinable function vectors with reflection property have good balancing orders. We refer to B. Han [4] for the approximation property and balancing order properties refinable function vectors with a general dilation factor have.

Since sampling theorems of wavelet applications in signal processing are desirable [5,6] and one can expect the equivalence of the approximation and balancing orders [7–9], we focus our attention to constructing compactly supported interpolating refinable function vectors having the approximation order property. Interesting results and examples of

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interpolating refinable function vectors and multiwavelets have been already known for certain dilation factors and multiplicities, which are univariate or multivariate [6,10,7,11,2,12,13].

In what follows, we consider refinable function vector $\boldsymbol{\Phi} = (\phi_1, \phi_2, \dots, \phi_r)^T$ having special reflection property with the dilation factor N and multiplicity r. For example, if r equals to 3, then $\phi_3(\cdot) = \phi_1(2\alpha - \cdot)$ and ϕ_2 is symmetric by itself with a suitable real constant α. Conditions for the filters to provide reflection pairs are investigated in this paper. A necessary condition in terms of coefficients of the mask is established, but unfortunately an interpolating refinable function vector that satisfies such condition does not necessarily constitute reflection pairs, which is illustrated by examples. Nonetheless, for certain pairs of N and r this condition is sufficient as well. On the other hand, it is also possible to construct a trivial solution that consists of translated versions of a main component function which is symmetric by itself. Our results are accompanied by relevant examples, each of which is compactly supported and satisfies a suitable approximation order (or sum rules). The regularity is checked by calculating Sobolev exponent as in [14,15].

This paper is organized as follows: in the following section, we introduce refinable function vectors and their refinement masks. Interpolating and approximation order properties are studied for any dilation factor and multiplicity. In Section 3, we redefine an interpolating refinable function vector having reflection property and give the general necessary condition for any dilation factor and multiplicity. And then, we investigate the existence of solutions for such refinement equation. Section 4 gives certain L^2-stable examples with some approximation orders. Additionally, a counterexample showing the insufficiency of the necessary condition for an interpolating refinable function vector to have the reflection property is provided and discuss repeating solutions in case $N = r = 3$.

2. Preliminaries

Let N and r be integers greater than 1. We say that $\boldsymbol{\Phi} = (\phi_1, \dots, \phi_r)^T$ is said to be a *refinable function vector* with the dilation factor N and the multiplicity r if $\boldsymbol{\Phi}$ satisfies a *matrix refinement equation*

$$\boldsymbol{\Phi}(x) = N \sum_{k\in\mathbb{Z}} \mathbf{P}_k \boldsymbol{\Phi}(Nx - k) \tag{1}$$

for some $r \times r$ real matrices $\mathbf{P}_k$.

The Fourier transform is defined by $\hat{\boldsymbol{\Phi}} := \left(\hat{\phi}_1, \dots, \hat{\phi}_r\right)^T$, where $\hat{\phi}_j(\omega) := \int_{-\infty}^{\infty} \phi_j(x) e^{-i\omega x} dx$ with $i = \sqrt{-1}$ for $j = 1, \dots, r$. In frequency variable, (1) can be written as

$$\hat{\boldsymbol{\Phi}}(\omega) = \mathbf{P}\left(\frac{\omega}{N}\right) \hat{\boldsymbol{\Phi}}\left(\frac{\omega}{N}\right), \tag{2}$$

where $\mathbf{P}(\omega)$ is the *matrix refinement mask* corresponding to $\boldsymbol{\Phi}$ defined by $\mathbf{P}(\omega) := \sum_{k\in\mathbb{Z}} \mathbf{P}_k e^{-i\omega k}$.

Assuming that $\boldsymbol{\Phi}$ is compactly supported and $\hat{\boldsymbol{\Phi}}(\omega)$ is continuous at 0 with $\hat{\boldsymbol{\Phi}}(0) \neq \mathbf{0}$, a nontrivial solution of (2) can exist only if $\mathbf{P}(0)$ has a simple eigenvalue 1 and $\hat{\boldsymbol{\Phi}}(0)$ is the corresponding eigenvector [16–19]. A refinable function vector $\boldsymbol{\Phi}$ in (1) can be found approximately by the cascade algorithm and the necessary and sufficient conditions are provided in [20]. If there exist constants $0 < A \le B < \infty$ such that

$$A \sum_{k=-\infty}^{\infty} \mathbf{b}_k^* \mathbf{b}_k \le \left\| \sum_{k=-\infty}^{\infty} \mathbf{b}_k^* \boldsymbol{\Phi}(\cdot - k) \right\|_{L^2}^2 \le B \sum_{k=-\infty}^{\infty} \mathbf{b}_k^* \mathbf{b}_k$$

holds for any vector sequence $\{\mathbf{b}_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z}^r)$, then $\boldsymbol{\Phi}$ is said to be L^2-*stable*. One can check the L^2-stability and smoothness of a refinable function vector as in [14,15]. We say that a function vector $\boldsymbol{\Phi} = (\phi_1, \dots, \phi_r)^T$ is *interpolating* if $\boldsymbol{\Phi}$ is continuous and

$$\phi_j\left(n + \frac{m-1}{r}\right) = \delta_{n,0}\delta_{j,m} \tag{3}$$

for $n \in \mathbb{Z}$ and $j, m = 1, \dots, r$. Let $\mathbf{P}(\omega)$ be the corresponding refinement mask of $\boldsymbol{\Phi}$ denoted by $[\mathbf{P}(\omega)]_{j,k} := p_{j,k}(\omega) = \sum_{\ell\in\mathbb{Z}} c_\ell^{j,k} e^{-i\omega\ell}$ with real constants $c_\ell^{j,k}$ for $j, k = 1, \dots, r$. The following lemma is due to B. Han et al. [2]. The proof is included for the reader's convenience.

Lemma 2.1. *If $\boldsymbol{\Phi}$ is an interpolating refinable function vector, then the corresponding coefficients satisfy for $n \in \mathbb{Z}$*

$$c_{Nn+q_m}^{j,t_m} = \frac{1}{N}\delta_{n,0}\delta_{j,m}, \tag{4}$$

where q_m and t_m are the integers such that

$$q_m := \left\lfloor \frac{N(m-1)}{r} \right\rfloor \quad \text{and} \quad t_m := N(m-1) - rq_m + 1 \in \{1, \dots, r\}$$

for $j, m = 1, \dots, r$.