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On one approach to solve the linear boundary value problems for Fredholm integro-differential equations

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ABSTRACT

Necessary and sufficient conditions for the well-posedness of linear boundary value problems for Fredholm integro-differential equations are established. Algorithms for finding their solutions are offered.

1. Introduction

Integro-differential equations frequently arise in applications being the mathematical models of some processes in physics, biology, chemistry, economy, etc. Their role in the study of processes with after effects was noted in the monograph [1], and the overview of early works devoted to the initial and boundary value problems for integro-differential equations was provided as well. For the various aspects of qualitative theory and approximate methods for integro-differential equations, and their applications we refer to [1–16].

Consider the linear boundary value problem for Fredholm integro-differential equation

$$\frac{dx}{dt} = A(t)x + \int_0^T K(t,s)x(s)ds + f(t), \quad t \in (0,T),\ x \in R^n, \tag{1.1}$$

$$Bx(0) + Cx(T) = d, \quad d \in R^n, \tag{1.2}$$

where the $(n \times n)$ matrices $A(t)$ and $K(t,s)$ are continuous on $[0,T]$ and $[0,T] \times [0,T]$, respectively; the n vector $f(t)$ is continuous on $[0,T]$.

A solution to problem (1.1), (1.2) is a vector function $x(t)$, continuous on $[0,T]$ and continuously differentiable on $(0,T)$. It satisfies the integro-differential equation (1.1) and boundary condition (1.2).

Reduction of problems for integro-differential equations to the integral equations is the main tool in investigating and solving the considered problems.

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In Nekrasov's method [17], the integral term of Eq. (1.1) is assumed as the right-hand side of corresponding differential equation. Using a fundamental system of solutions to this equation, Eq. (1.1) is reduced to the Fredholm integral equation of second kind. If the latter one is uniquely solvable, then the general solution to Fredholm integro-differential equation can be written down via the resolvent of integral equation. Now, the solvability of problem (1.1), (1.2) is equivalent to the solvability of linear system of algebraic equations compiled by the general solution and boundary condition (1.2).

Green's function method is another commonly used one. In this method, the integral term also refers to the right-hand side of differential equation. Under assumption on unique solvability of boundary value problem for the differential equation we construct its Green's function and then reduce the origin boundary value problem for Fredholm integro-differential equation to the Fredholm integral equation of second kind. Solving this equation, we find the desired function.

The mentioned methods are a base for the approximate methods considered in [7]. Under assumption on the solvability of Cauchy and boundary value problems for integro-differential equations, their approximate solutions are found by the method of averaging functional corrections, method of continuation on parameters, methods of S.A. Chaplygin, B.G. Galerkin, etc. Expanding the application areas of integro-differential equations and development of approximate methods for finding their solutions have led to the modification of known methods and creation of new ones. Overview of some modern and widely used approximate methods is contained in the monograph [16]. Effectiveness of direct computing method and method of successive approximations, variational-iterative method, decomposition method, method of series and other methods is illustrated by solving the problems for integro-differential equations. Conditions for the existence of solution to the considered problems are not given in this monograph. This is typical for the most of works devoted to the approximate methods for finding the solutions to initial and boundary value problems for integro-differential equations. Under assumption on the solvability of problems they are focused on the algorithms for finding their approximate solutions.

However, the Fredholm integro-differential equation has a row of features that must be taken into consideration while creating the approximate methods for finding its solutions.

Let us consider the equation

$$\frac{dx}{dt} = Ax + \frac{1}{2\pi}\int_0^{2\pi} Bx(s)ds + f, \quad t \in (0, 2\pi),\ x \in R^2,$$

where

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

As follows from the results of [2, p. 71] this Fredholm integro-differential equation has no solution. At the same time the linear ordinary differential equations and linear Volterra integro-differential equations are solvable for any right-hand side f, and they have a family of solutions. For the linear ordinary differential equation on $[a, b]$ the Cauchy problem with initial condition given at any point of this interval has a unique solution. If we consider the linear Volterra integro-differential equation on $[a, b]$, and the left-end of interval is the lower limit of integral term, then the Cauchy problem for this equation with condition at the point a is also uniquely solvable. Therefore, the Cauchy problem is often used as an intermediate problem under investigating the qualitative properties of boundary value problems for the ordinary differential equations and Volterra integro-differential equations.

However, the Cauchy problem for the Fredholm integro-differential equation can be unsolvable, but the boundary value problem for this equation may have a unique solution. Indeed, if we suppose that a function $x^*(t)$ satisfies the integro-differential equation

$$\frac{dx}{dt} = 2\int_0^1 x(s)ds + 2t - \frac{2}{3}, \quad t \in (0, 1),\ x \in R^1$$

and the initial condition $x(0) = 1$, then after certain easy calculations we come to the contradiction: $-1 = 0$. As it was shown in [18, p. 1160], the boundary value problem for this equation with the condition $x(0) = x(1)$ has a unique solution $x(t) = t(t - 1)$.

These features of Fredholm integro-differential equations require an examination of qualitative properties of the problem (1.1), (1.2), before starting to construct the approximate methods for its solving.

The aim of this paper is to bring together the criteria of well-posedness of linear boundary value problems for Fredholm integro-differential equations and the algorithms, including numerical, for finding their solutions.

To reach the goal we use the parameterization method [19]. Method is based on dividing the interval $[0, T]$ into N parts and introducing the additional parameters. While applying the method to problem (1.1), (1.2), the necessity of solving an intermediate problem also arises. Intermediate problem here is the special Cauchy problem for the system of integro-differential equations with parameters. But unlike the intermediate problems of above mentioned methods, the special Cauchy problem is always uniquely solvable for sufficiently small partition step. This property of intermediate problem allowed to establish in [18] the necessary and sufficient conditions for solvability and unique solvability of problem (1.1), (1.2).

In [20,21] the smallness of interval's partition step is also required in the algorithms for solving the linear boundary value problems for Fredholm integro-differential equations.