Journal of Computational and Applied Mathematics 295 (2016) 25–34

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Asymptotic expansions for high-contrast linear elasticity

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ARTICLE INFO

Article history:
Received 10 October 2014
Received in revised form 6 March 2015

Keywords:
Linear elasticity problem
High-contrast coefficients
Asymptotic expansions
Highly inelastic inclusion
Convergence

ABSTRACT

We study linear elasticity problems with high contrast in the coefficients using asymptotic limits recently introduced. We derive an asymptotic expansion to solve heterogeneous elasticity problems in terms of the contrast in the coefficients. We study the convergence of the expansion in the H^1 norm.

1. Introduction

There is a growing interest in the computation of solutions of problems governed by partial differential equations with high-contrast coefficients. Solutions to these model problems are multiscale in nature. The solutions to these problems are often approximated using the Finite Element Method (FEM), Multiscale Finite Element Method (MsFEM) or alternative forms of these, (cf., [1–6] and references therein).

Herein, we study linear elasticity problems in heterogeneous media. Our goal is to devise approximate solutions that account for the high contrast in the coefficients. We focus on the dependence of the contrast in the coefficients where the contrast is referred to as the ratio of the jumps in the physical properties. We follow the analysis presented in [7] that consists of deriving an asymptotic expansion for the solution of the elliptic differential equation in heterogeneous media. Thus, we derive asymptotic expansions to solve linear elasticity problems with high contrast.

The linear elasticity equations model the equilibrium and the local strain of deformable bodies; see [8–12]. The constitutive laws relating stresses and strains depend on the material and the process modeled. For composite materials, physical properties such as Young's modulus can vary several orders of magnitude and we seek to understand the effects of these variations on the solution. In this setting, the asymptotic expansions that express the solution are useful tools to understand the effects of the high contrast and the interactions between different materials.

We consider the equilibrium equations for a linear elastic material in a smooth domain $D \subset \mathbb{R}^d$. Given $u \in H^1(D)^d$ that represents the displacement field, we denote

$$\epsilon = \epsilon(u) = \left[\epsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right],$$

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http://dx.doi.org/10.1016/j.cam.2015.03.021
0377-0427/

where ϵ is the strain tensor which linearly depends on the derivatives of the displacement field u (see [13,14]). We also introduce the stress tensor $\tau(u)$, which depends on the value of strains and is defined as

$$\tau = \tau(u) = 2\mu\epsilon(u) + \lambda \mathrm{tr}\epsilon(u) I_{d\times d}, \tag{1}$$

where $I_{d\times d}$ is the identity matrix in $\mathbb{R}^d$ and $\mathrm{tr}\epsilon(u) = \mathrm{div}(u)$. The Lamé coefficients λ and μ describe the elastic response of an isotropic material, see, e.g., [13,12].

We assume that the Poisson ratio $\nu = \frac{\lambda}{2(\lambda+\mu)}$ is bounded away from 0.5, i.e., the Poisson ratio satisfies $0 < \nu \le \nu_0 < 0.5$ for some constant value ν_0. The volumetric strain modulus is given by

$$K = \frac{E}{3(1-2\nu)} > 0,$$

and thus $1 - 2\nu > 0$ (see [12]). Given these assumptions, then $\nu = \nu(x)$ can only have mild variations in D.

We introduce the heterogeneous function $E = E(x)$ that represents Young's modulus and thus express the shear modulus as

$$\mu(x) = \frac{1}{2}\frac{E(x)}{1+\nu(x)} = \tilde{\mu}(x)E(x),$$

where $\tilde{\mu} = 1/2(1+\nu)$. Thus,

$$\lambda(x) = \frac{1}{2}\frac{E(x)\nu(x)}{(1+\nu(x))(1-2\nu(x))} = \tilde{\lambda}(x)E(x),$$

where we use $\tilde{\lambda} = \frac{\nu}{2(1+\nu)(1-2\nu)}$. The spatial variation of E drives the multiscale response of the solution. We denote

$$\tilde{\tau}(u) = 2\tilde{\mu}\epsilon(u) + \tilde{\lambda}\mathrm{tr}\epsilon(u) I_{d\times d}.$$

Given a vector field f we consider the problem

$$-\mathrm{div}(\tau(u)) = f, \quad \text{in } D, \tag{2}$$

with $u = g$ on ∂D. The tensor τ is defined in (1). We analyze in detail a binary medium $E(x)$ with elastic background and one inclusion (a stiff body) for the case of one inelastic inclusion. The analysis of the case with several highly inelastic inclusions is similar. To parametrize the problem, we consider the background with stiffness 1 and the inclusions with a relative stiffness denoted by η. We derive expansions of the form (see [7])

$$u_\eta = u_0 + \frac{1}{\eta}u_1 + \frac{1}{\eta^2}u_2 + \cdots. \tag{3}$$

We define each term in the expansion using local problems. We then study the convergence of the expansion in the H^1 norm.

The rest of the paper is organized as follows. In Section 2 we recall the weak formulation and provide an overview of the derivation of the expansion for high contrast inclusions. In Section 3, the convergence for this asymptotic expansion is described. Finally, in Section 4 we state our conclusions and final comments.

2. One interior inclusion problem: problem statement

Let $D \subset \mathbb{R}^d$ be a polygonal domain or a domain with smooth boundary. We consider the following weak formulation of (2). Find $u \in H^1(D)^d$ such that

$$\begin{cases} \mathcal{A}(u, v) = \mathcal{F}(v), & \text{for all } v \in H^1_0(D)^d, \\ u = g, & \text{on } \partial D, \end{cases} \tag{4}$$

where the bilinear form $\mathcal{A}$ and the linear functional $\mathcal{F}$ are defined by

$$\mathcal{A}(u, v) = \int_D 2\tilde{\mu}E\epsilon(u)\cdot\epsilon(v) + \tilde{\lambda}E\mathrm{tr}\epsilon(u)\mathrm{tr}\epsilon(v), \quad \text{for all } u, v \in H^1_0(D)^d, \tag{5}$$

and

$$\mathcal{F}(v) = \int_D fv, \quad \text{for all } v \in H^1_0(D)^d, \tag{6}$$

respectively, with $\epsilon(u)\cdot\epsilon(v) := \sum_{i,j=1}^d \epsilon_{ij}(u)\epsilon_{ij}(v)$.

The domain D is the disjoint union of a background domain and one inclusion, that is, $D = D_0 \cup \overline{D}_1$. We assume that D_0 and D_1 are polygonal domains or domains with smooth boundaries. Let D_0 represent the background domain and the