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An iterative numerical method for singularly perturbed reaction–diffusion equations with negative shift

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ABSTRACT

In this paper, a numerical method based on an iterative scheme is proposed for a singularly perturbed delay differential equation of reaction–diffusion type. In this method the solution of the delay problem is obtained as the limit of the solutions to a sequence of the non-delay problems. The numerical solutions of the non-delay problems are obtained by applying existing finite difference schemes and finite element method for the non-delay singularly perturbed equations. Error estimates are derived in the supremum norm. Numerical results illustrating the theory are also included.

1. Introduction

Delay differential equations appear in the mathematical modeling of various practical phenomena, where they provide the best approximations of the observed phenomena. Problems of this type occur where the future depends not only on the immediate present, but also on the past history of the system under consideration. A delay differential equation is of the retarded type if the delay argument does not occur in the higher order derivative term. If we restrict the class in which the highest derivative is multiplied by a small parameter, then we get singularly perturbed delay differential equations of retarded type. Such type of equations arises frequently in the mathematical modeling of various practical phenomena, for example, in the modeling of the human pupil-light reflex [1], the study of bistable devices [2] and variational problems in control theory [3], etc. The computation of the solutions of these delay equations has been a great challenge for several years.

It is a well known fact that the standard discretization methods for solving singular perturbation problems for differential equations are sometimes unstable and fail to give accurate results when the perturbation parameter ε is small. Therefore, it is important to develop suitable numerical methods to solve this type of equations, whose accuracy do not depend on the parameter ε, that is methods which are uniformly convergent with respect to the parameter ε.

In the past, only very few people worked in the area Numerical methods to Singularly Perturbed Delay Differential Equations (SPDDEs). But in recent years, there has been a growing interest in this area. In fact, Fevzi Erdogan [4] proposed an exponentially fitted operator method for singularly perturbed first order delay differential equations, Kadalbajoo and Sharma [5–7] and Jugal Mohapatra and Natesan [8] proposed few numerical methods for SPDDEs with small delays. Subburayan and Ramanujam [9–14] suggested numerical methods named as initial value technique and asymptotic numerical method for singularly perturbed delay differential equations of reaction–diffusion type as well as convection–diffusion type.

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Serge Nicaise and Christos Xenophontos [15], developed a *hp*-version finite element method to find an approximation to a solution of the singularly perturbed second order differential equation with a constant delay. In [16], the author proposed a discontinuous Galerkin finite element method for a singularly perturbed delay differential equation of reaction–diffusion type and established robust convergence of the method in the energy norm.

Using the iterative procedure suggested in [17] for a second order ordinary delay differential equation and finite difference schemes and a finite element method available in the literature for non-delay singularly perturbed differential equations, we, in this paper, propose an iterative method to find a numerical solution for the following singularly perturbed delay differential equations of reaction–diffusion type.

Find $u \in U := C^2(\Omega) \cap C(\overline{\Omega})$ such that

$$\begin{cases} -\varepsilon u''(x) + a(x)u(x) + b(x)u(x-1) = f(x), & x \in \Omega, \\ u(x) = \phi(x), & x \in [-1, 0], \\ u(2) = l, \end{cases} \tag{1.1}$$

where $0 < \varepsilon \ll 1$, $0 < \alpha \le a(x)$, $-\beta_0 \le b(x) \le \beta < 0$, for all $x \in \overline{\Omega}$, $\Omega = (0, 2)$, $\overline{\Omega} = [0, 2]$, $\alpha - \beta_0 > 0$, the functions a, b and $f \in C^4(\overline{\Omega})$ and $\phi \in C^4([-1, 0])$.

The above boundary value problem (1.1) has a solution and the solution is unique [17].

The present paper is organized as follows. Section 2 presents the maximum principle and the stability result. The proposed iterative procedure is explained in Section 3. Mesh selection strategy is discussed in Section 4. First and higher order finite difference schemes are presented in Sections 5 and 6 respectively, whereas Section 7 deals with the finite element method. Numerical experiments are carried out in Section 8. The paper concludes with a discussion.

Throughout our analysis C is a generic positive constant that is independent of parameter ε and number of mesh points N. Further, the supremum norm used for studying the convergence of the numerical solution to the exact solution of a singular perturbation problem is

$$\|u\|_D = \sup_{x \in D} |u(x)|.$$

2. Maximum principle and stability result

Consider the following problem.

Find $u \in U^* := C^2(\Omega^- \cup \Omega^+) \cap C(\overline{\Omega})$ such that

$$\begin{cases} Pu(x) := \begin{cases} -\varepsilon u''(x) + a(x)u(x) = f(x) - b(x)\phi(x-1), & x \in \Omega^-, \\ -\varepsilon u''(x) + a(x)u(x) + b(x)u(x-1) = f(x), & x \in \Omega^+, \end{cases} \\ u(0) = \phi(0), \quad u(1^-) = u(1^+), \quad u'(1^-) = u'(1^+), \quad u(2) = l, \end{cases} \tag{2.1}$$

where $\Omega^- = (0, 1)$ and $\Omega^+ = (1, 2)$. This boundary value problem (2.1) exhibits strong boundary layers at $x = 0,\ x = 2$ and strong interior layers (left and right) at $x = 1$ [18].

The above differential operator P satisfies the following maximum principle.

Theorem 2.1 (*Maximum Principle*). *Let $w \in U^*$ be any function satisfying $w(0) \ge 0,\ w(2) \ge 0,\ Pw(x) \ge 0,\ \forall x \in \Omega^- \cup \Omega^+$ and $[w'](1) \le 0$, where $[w'](1) = w'(1^+) - w'(1^-)$. Then $w(x) \ge 0,\ \forall x \in \overline{\Omega}$.*

Proof. Using the basic idea used in the proof of Theorem 3.1 of [14] and the test function $s(x)$ given by,

$$s(x) = \begin{cases} \dfrac{1}{8} + \dfrac{x}{2}, & x \in [0, 1], \\ \dfrac{3}{8} + \dfrac{x}{4}, & x \in [1, 2], \end{cases}$$

the above theorem can be proved. □

An immediate consequence of the maximum principle is the following stability result.

Corollary 2.2 (*Stability Result*). *For any $w \in U^*$, we have*

$$|w(x)| \le C \max\left\{ |w(0)|,\ |w(2)|,\ \sup_{\xi \in \Omega^- \cup \Omega^+} |Pw(\xi)| \right\}, \quad \forall\, x \in \overline{\Omega},$$

where $C = 8 \max\left\{1,\ \frac{1}{\alpha},\ \frac{1}{5\alpha - 5\beta_0}\right\}$.