Journal of Computational and Applied Mathematics 296 (2016) 56–88

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Robustness of quadratic hedging strategies in finance via Fourier transforms

Catherine Daveloose [a,*], Asma Khedher [b], Michèle Vanmaele [a]

[a] Department of Applied Mathematics, Computer Science and Statistics, Ghent University, Krijgslaan 281-S9, 9000 Gent, Belgium

[b] Korteweg–de Vries Institute for Mathematics, University of Amsterdam, P.O. Box 94248, 1090 GE Amsterdam, The Netherlands

ARTICLE INFO

Article history:
Received 26 September 2014
Received in revised form 16 July 2015

Keywords:
Lévy processes
Options
Quadratic hedging
Fourier transforms
Robustness

ABSTRACT

In this paper we investigate the consequences of the choice of the model to partial hedging in incomplete markets in finance. In fact we consider two models for the stock price process. The first model is a geometric Lévy process in which the small jumps might have infinite activity. The second model is a geometric Lévy process where the small jumps are truncated or replaced by a Brownian motion which is appropriately scaled. To prove the robustness of the quadratic hedging strategies we use pricing and hedging formulas based on Fourier transform techniques. We compute convergence rates and motivate the applicability of our results with examples.

1. Introduction

In financial markets, the hedging of derivatives is in general set in the non-arbitrage framework and can technically be performed under a related pricing measure that is a risk-neutral measure. Under this measure the discounted prices of the underlying primaries are martingales. In some markets, for example, in the context of energy derivatives, the underlying, electricity, cannot be stored. Hence hedging does not require that the pricing measure is a risk-neutral measure. See e.g. [1] for more details. In this case the discounted stock price process is a semimartingale under the pricing measure.

To model asset price dynamics we consider two geometric Lévy processes. This type of models describe well realistic asset price dynamics and are well established in the literature (see e.g. [2]). The first model $(S_t)_{t\in[0,T]}$ is driven by a Lévy process in which the small jumps might have infinite activity. The second model $(S_t^\varepsilon)_{t\in[0,T]}$ is driven by a Lévy process in which we truncate the jumps with absolute size smaller than $\varepsilon > 0$ or we replace them by an appropriately scaled Brownian motion. That is

$$S_t^\varepsilon = S_0 \exp(N_t^\varepsilon + s(\varepsilon)\widetilde{W}_t), \tag{1.1}$$

where $S_0 > 0$ is the initial price process, $(N_t^\varepsilon)_{t\in[0,T]}$ is a Lévy process with jumps bigger than ε, $(\widetilde{W}_t)_{t\in[0,T]}$ is an independent Brownian motion and $0 \le s(\varepsilon) \le 1$. The scaling factor $s(\varepsilon)$ should be either equal to zero or any sequence which vanishes when ε goes to 0. Notice that in this case, $(S_t^\varepsilon)_{t\in[0,T]}$ approximates $(S_t)_{t\in[0,T]}$. We do not discuss any preferences for the choice of $s(\varepsilon)$. We only present different possible choices and exploit the consequences of the approximation on the pricing and hedging formulas.

Because of the presence of jumps, the market is in general incomplete and there is no self- financing hedging strategy which allows to attain the contingent claim at maturity. In other words, one cannot eliminate the risk completely. The most

* Corresponding author.
E-mail addresses: Catherine.Daveloose@UGent.be (C. Daveloose), A.Khedher@UvA.nl (A. Khedher), Michele.Vanmaele@UGent.be (M. Vanmaele).

http://dx.doi.org/10.1016/j.cam.2015.09.005
0377-0427/

common approach to hedge financial risks is to implement a dynamic delta-hedging strategy. In case of frequent rebalancing such strategy can be too costly because of transaction costs. Static hedging strategies on the contrary take a position in a portfolio of financial instruments that are traded in the market (at least over-the-counter) and have no intermediate costs between the set up of the strategy and the maturity of the financial derivative. The portfolio is chosen such that the hedge matches as well as possible the payoff of the financial claim that is hedged. Super-replicating static hedging strategies are on the safe side and will end up at maturity with a portfolio value which is larger than the payoff, but can be very expensive. In the literature also a combination is suggested, namely a semi-static hedging strategy. The hedging portfolio is constructed at each rebalancing date by following an optimal hedging strategy for some optimality criterion that is not altered until the next rebalancing date.

The present paper is concerned with the notion of 'partial' hedging strategies which minimise some risk. One way to determine these 'partial' hedging strategies is to introduce a subjective criterion according to which strategies are optimised. These 'partial' hedging strategies can be used in a dynamic way or a semi-static way. We consider two types of quadratic hedging. In the first approach, called *mean–variance hedging* (MVH), the strategy is self-financing and one minimises the quadratic hedging error at maturity in mean square sense (see, e.g., [3]). The second approach is called *risk-minimisation* (RM) in the martingale setting and *local risk-minimisation* (LRM) in the semimartingale setting. These strategies replicate the option's payoff, but they are not self-financing (see, e.g., [3]). In the martingale setting the RM strategies minimise the risk process which is induced by the fact that the strategy is not self-financing. In the semimartingale setting the LRM strategies minimise the risk in a 'local' sense (see [4,5]).

We study the robustness of hedging strategies to model risk. As sources to model risk one may think of the errors in estimations of the parameters or in market data, or errors resulting from discretisation. In this paper, we focus on the *risk to model choice*. In this context we may think of two financial agents who want to price and hedge an option. One is considering $(S_t)_{t\in[0,T]}$ as a model for the underlying stock price process and the other is considering $(S_t^\varepsilon)_{t\in[0,T]}$. In other words, the first agent chooses to consider infinitely small variations in a discontinuous way, i.e. in the form of infinitely small jumps of an infinite activity Lévy process. The second agent observes the small variations in a continuous way, i.e. coming from a Brownian motion. Hence the difference between both market models determines a type of model risk. This is a relevant study from practical point of view. Indeed if two strategies are different but lead to the same hedging error, they are equivalent from the point of view of risk management.

In the sequel, we intend by *robustness* of the model, the convergence results when ε goes to zero of $(S_t^\varepsilon)_{t\in[0,T]}$ and of its related pricing and hedging formulas. In [6], it was proved that the convergence of asset prices does not necessarily imply the convergence of option prices and found out that pricing and hedging are in general not robust. However, in [7] the approximation of the form (1.1) inspired by [8], was investigated and it was proved that the related option prices and the deltas are robust. In this paper we investigate whether the corresponding quadratic hedging strategies are also robust and we reconsider the conditions obtained in [7].

For the study of robustness, we first use Fourier transform techniques as in [9,10]. In these two papers, the authors considered the case where the market is observed under a martingale measure and wrote the option prices and hedging strategies for European options in terms of the Fourier transform of the *dampened* payoff function and the characteristic function of the driving Lévy process. We use these formulas when the market is considered under each of the following equivalent martingale measures: the Esscher transform (ET), the minimal entropy martingale measure (MEMM), the minimal martingale measure (MMM), and the variance-optimal martingale measure (VOMM). In this context and under some integrability conditions on the Lévy process and the payoff function, we prove the robustness of the *optimal number* of risky assets in a RM strategy as well as in a MVH strategy. Moreover we compute convergence rates for our results.

Secondly, in case the market is observed under the world measure and thus the discounted stock price processes are modelled by semimartingales, the hedging strategies are written in [11] in terms of the cumulant generating function of the Lévy process and a complex measure which depends on the Fourier transform of the dampened payoff function. In this setting and under certain integrability conditions on the Lévy process and the payoff function, we also prove the robustness of the *optimal number* of risky assets in a LRM strategy. Moreover we prove the robustness of the *amount of wealth* in the risky asset in a MVH strategy and we compute convergence rates.

The approach in this paper allows us to continue and further analyse the results studied in [12]. In the latter paper, robustness results of the amount of wealth in LRM and MVH strategies were studied for a more general asset price setting in the semimartingale case. Thereto the authors used backward-stochastic differential equations. In the present paper we prove in addition the robustness of the *number of risky assets* in RM and LRM strategies and we discuss the robustness of different equivalent martingale measures. Notice that the martingale measure depends on the model chosen for the asset price. Thus for this study we need to specify the model. Moreover, the use of the Fourier approach allows us to compute convergence rates. Finally, to illustrate our study we give examples of payoff functions and Lévy processes that fulfil the conditions we impose for the convergence.

The present paper is a good survey of the use of Fourier transform techniques in quadratic hedging strategies. In fact, we present the different methods existing in the literature and we discuss the applicability of these methods complemented with examples.

The paper is organised as follows. In Section 2 we introduce the notations and recall some recent results about hedging in incomplete markets. In Section 3 we investigate the robustness of the quadratic hedging strategies where the stock price processes are observed under martingale measures. In Section 4 we prove the robustness of the quadratic hedging