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The Euler–Galerkin finite element method for a nonlocal coupled system of reaction–diffusion type

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ABSTRACT

In this work, we study a system of parabolic equations with nonlocal nonlinearity of the following type

$$\begin{cases} u_t - a_1(l_1(u), l_2(v))\Delta u + \lambda_1|u|^{p-2}u = f_1(x,t) & \text{in } \Omega\times]0,T] \\ v_t - a_2(l_1(u), l_2(v))\Delta v + \lambda_2|v|^{p-2}v = f_2(x,t) & \text{in } \Omega\times]0,T] \\ u(x,t) = v(x,t) = 0 & \text{on } \partial\Omega\times]0,T] \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x) & \text{in } \Omega \end{cases},$$

where a_1 and a_2 are Lipschitz-continuous positive functions, l_1 and l_2 are continuous linear forms, $\lambda_1, \lambda_2 \geq 0$ and $p \geq 2$.

We prove the convergence of a linearized Euler–Galerkin finite element method and obtain the order of convergence in the L_2 norm. Finally we implement and simulate the presented method in Matlab's environment.

1. Introduction

The reaction–diffusion equation can describe physical, chemical, biological and ecological systems.

In 1997, M. Chipot and B. Lovat [1] studied the nonlocal problem

$$\begin{cases} u_t - a(l(u))\Delta u = f(x,t) & \text{in } \Omega\times(0,T) \\ u(x,t) = 0 & \text{on } \partial\Omega\times(0,T) \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1}$$

where Ω is a bounded open subset in $\mathbb{R}^N$, $N \geq 1$, with smooth boundary $\partial\Omega$, T is some arbitrary time and a is some function from $\mathbb{R}$ into $(0,+\infty)$. In problem (1), a and f are both continuous functions and $l : L^2(\Omega) \to \mathbb{R}$ is a continuous linear form. This problem arises in various situations, for instance, u could describe the density of a population (for example,

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of bacteria) subject to spreading. If we assume that the movements are guided by the global state of the medium, then the diffusion coefficient a must depend on the entire population in the domain rather than on the local density. This model also appears in the study of heat propagation in a conductor, considering the fact that the measurements are not made at a point but represent an average in a neighborhood. The problem studied is nonlocal in the sense that the diffusion coefficient is determined by a global quantity. The above-mentioned authors proved the existence and uniqueness of solutions to this problem.

In 2004, Corrêa, Menezes and Ferreira [2] gave an extension of the result obtained by M. Chipot and B. Lovat [1], considering $a = a(l(u))$ and $f = f(x, u)$ continuous functions.

In 2000, Ackleh and Ke [3] studied the problem

$$\begin{cases} u_t = \dfrac{1}{a\left(\int_\Omega u\,dx\right)}\Delta u + f(u) & \text{in } \Omega\times]0,T] \\ u(x,t) = 0 & \text{on } \partial\Omega\times]0,T] \\ u(x,0) = u_0(x) & \text{on } \overline{\Omega}, \end{cases}$$

with $a(\xi) > 0$ for all $\xi \neq 0$, $a(0) \geq 0$ and f Lipschitz-continuous satisfying $f(0) = 0$. They proved the existence and uniqueness of solutions to this problem and gave conditions on u_0 for the extinction in finite time and for the persistence of solutions. In that work they propose a finite difference scheme to approximate the solutions and to study their long time behavior.

Raposo et al. [4], in 2008, studied the reaction–diffusion coupled system in a parallel way, via a parameter $\alpha = \text{const} > 0$, of the form

$$\begin{cases} u_t - a(l(u))\Delta u + f(u-v) = \alpha(u-v) & \text{in } \Omega\times]0,T] \\ v_t - a(l(v))\Delta v - f(u-v) = \alpha(v-u) & \text{in } \Omega\times]0,T], \end{cases}$$

with $a(\xi) > 0$, f a Lipschitz-continuous function and l a continuous linear form. They proved the existence, uniqueness and exponential decay of solutions. In [4] the authors also made numerical simulations, where they used an implicit finite difference scheme in one dimension and finite volume discretization [5] in two space dimensions. Simsen and Ferreira [6] studied the reaction–diffusion problem

$$\begin{cases} u_t - a(l(u))\Delta u + |u|^{p-2}u = f(u) & \text{in } \Omega \times (0,T) \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases}$$

In particular they investigated the existence, uniqueness, continuity with respect to the initial values, exponential stability of the weak solutions, continuity of the solution and an important result on the existence of the global attractor.

Recently, Duque et al. [7] considered a nonlinear coupled system of reaction–diffusion on a bounded domain with a more general nonlocal diffusion term acting on two linear forms l_1 and l_2:

$$\begin{cases} u_t - a_1(l_1(u), l_2(v))\Delta u + \lambda_1|u|^{p-2}u = f_1(x,t) & \text{in } \Omega\times]0,T] \\ v_t - a_2(l_1(u), l_2(v))\Delta v + \lambda_2|v|^{p-2}v = f_2(x,t) & \text{in } \Omega\times]0,T]. \end{cases} \tag{2}$$

In this case, u and v can describe the densities of two populations that interact through the functions a_1 and a_2. The death in species u is assumed to be proportional to $|u|^{p-2}u$ by the factor $\lambda_1 \geq 0$ and in species v to be proportional to $|v|^{p-2}v$ by the factor $\lambda_2 \geq 0$ with $p \geq 1$. The supply of being by external sources is denoted by f_1 and f_2. The authors improved the results obtained by Chipot and Lovat [1], Corrêa, Menezes and Ferreira [2], Raposo et al. [4] and Simsen and Ferreira [6] for coupled systems. They proved existence and uniqueness of weak and strong solutions global in time, and gave conditions on the data so that these solutions have the waiting time and stable localization properties. Moreover, important results on polynomial and exponential decay and vanishing of the solutions in finite time were also presented.

There are few studies on numerical approximations of nonlocal problems and these are quite restricted to nonlocal reaction terms or nonlocal boundary conditions. Yin and Xu [8] applied the finite volume element method to approximate solutions of a nonlocal problem on reactive flows in porous media and derived optimal convergence order in the L_2 norm. Sidi Ammi and Torres [9] proposed the application of the finite element method for the space variables and the Euler or Crank–Nicolson method for the time to fully discretize a nonlocal problem resulting from a thermistor problem. Moreover, they proved optimal rates of convergence in the L_2 norm.

This paper is concerned with the proof of the convergence in the L_2 norm of a total discrete solution using the Euler–Galerkin finite element method. To the best of our knowledge, these results are new for nonlocal reaction–diffusion coupled systems. This paper is organized as follows. In Section 2, we formulate the problem and the hypotheses on the data. In Section 3, we define and prove the convergence of the semidiscrete solution. Section 4 is devoted to the proof of the convergence to a fully discrete solution. Finally, in Section 5, we present some examples which illustrate an application of this theory.