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Logarithmic spirals and continue triangles

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ABSTRACT

In this article we will use some special triangles, to construct polygonal chains that describe the families of logarithmic spirals, among which are the celebrated Golden Spiral, Spira solaris and Pheidia Spiral.

1. Introduction

A *logarithmic spiral* is a plain curve whose equation in polar coordinate (ρ, θ) is $\rho = te^{(h\theta)}$. The term h is a positive number called the *growth constant* of the spiral, and t is a *constant of the spiral* depending on the choice of the initial condition $\theta = 0$. Note that θ increases anti-clockwise. The Cartesian representation of a logarithmic spiral is

$$\begin{cases} x(\theta) = \rho(\theta)\cos(\theta) = te^{(h\theta)}\cos(\theta) \\ y(\theta) = \rho(\theta)\sin(\theta) = te^{(h\theta)}\mathrm{sen}(\theta), \end{cases} \tag{1.1}$$

thus the distance from the origin (Pole) of $(x(\theta), y(\theta))$ increases exponentially when θ increases (anti-clockwise). Sometimes this kind of spiral is more precisely called a *Left hand logarithmic spiral*, to distinguish it from a *Right hand logarithmic spiral*, whose equation is of the type $\rho = te^{(-h\theta)}$. For the latter type of spirals the distance from the Pole of $(x(\theta), y(\theta))$ decreases exponentially when θ increases.

The most celebrated curve of this type is certainly the *Golden Spiral*, which is a logarithmic spiral whose growth constant is $(2/\pi)\lg(\Phi)$, where Φ is the "Golden Mean".

Looking at the equation of the Golden Spiral

$$\rho = e^{(2/\pi)\lg(\Phi)\theta} \quad \text{with starting point } (1, 0),$$

we note that for $\theta = 0$ we have $\rho = 1$, and for $\theta = \pi/2$ we have $\rho = \Phi$. More generally it can be easily seen that, a golden spiral gets wider (or further from its origin) by a factor of Φ for every quarter turn it makes; therefore "Φ^4" gives a measurement of the *growth factor* of this spiral after a complete turn around the pole.

The Golden Spiral was first described by Descartes, and then studied by the Swiss mathematician Jakob Bernoulli (1654–1705), who called it *Spira mirabilis*, and dedicated to it the famous motto "*Eadem Mutata resurgo*" which is inscribed on his tombstone.

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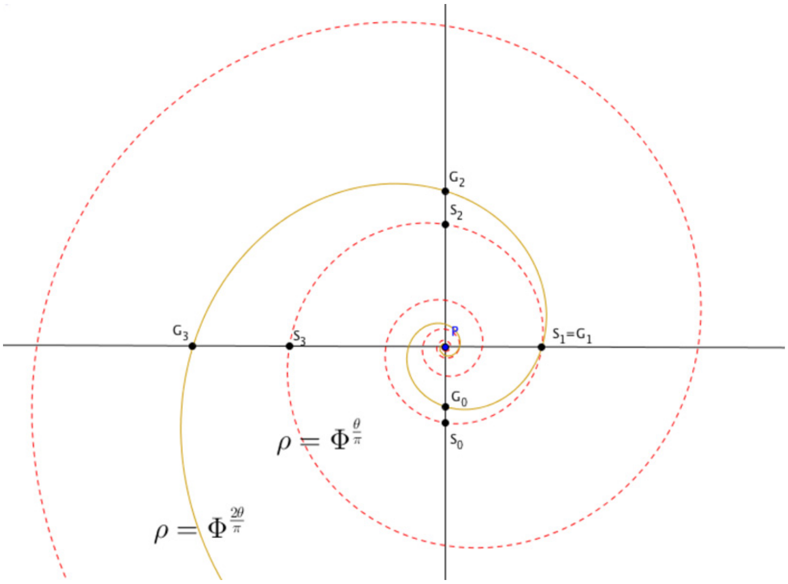

Fig. 1.1. Dotted Red line is the Spira Solaris, Gold line is the Golden Spiral.

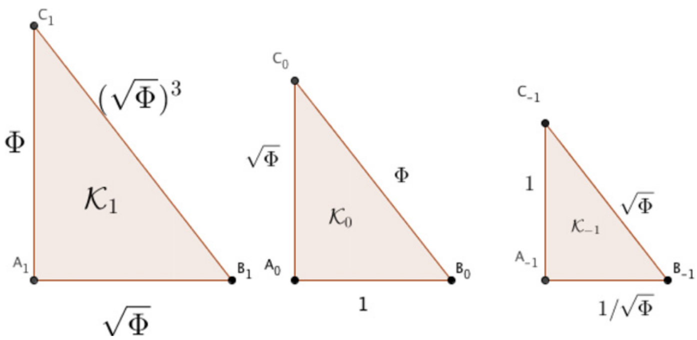

Fig. 1.2. Examples of Kepler triangles.

Approximations of logarithmic spirals can occur in nature (for example, the arms of spiral galaxies or phyllotaxis of leaves); golden spirals are one special case of these. It is sometimes stated that spiral galaxies and nautilus shells get wider in the pattern of a golden spiral, and hence are related to both Φ and the Fibonacci series. In truth, spiral galaxies and nautilus shells (and many mollusk shells) exhibit logarithmic spiral growth, but at a variety of angles usually distinctly different from that of the golden spiral. This pattern allows the organism to grow without changing shape (see [1,2]).

Also, we note a very interesting connection between the *DNA*-spiral and some Fibonacci-like sequences (see [3]). This highlights the relevant connection between recursive sequences and spirals (see also [4–7] and the references therein).

The German Mathematician Johannes Kepler (1571–1630) was the first to study the nature of the logarithmic spirals, and its possible *discretizations*. He was also attracted by their shape and their applications in astronomy. For this purpose he considered a logarithmic spiral, with a lower factor growth (Φ^2): The *Spira Solaris*, whose equation is given by (see Fig. 1.1):

$$\rho = e^{(1/\pi)\lg(\Phi)\theta} = \Phi^{\theta/\pi} \quad \text{with starting point } (1, 0).$$

Another celebrated logarithmic spiral is the *Pheidia Spiral*, whose equation is:

$$\rho = e^{(1/2\pi)\lg(\Phi)\theta} = \Phi^{\theta/2\pi}, \quad \text{with starting Point } (1, 0).$$

Note that Pheidia Spirals, Spira Solaris, and Golden Spiral have respectively the following "growth": Φ, Φ^2, Φ^4.

Connected to the Spira Solaris, there are the *Golden right triangle* or *Kepler triangle*, that is any right triangle $\mathcal{K}$ the lengths of whose sides, $a > b > c$ satisfy the proportion: $a{:}b = b{:}c$. Fig. 1.2 shows examples of Kepler triangles. For every integer n, we will denote by $\mathcal{K}_n$ a triangle the measurement of whose sides are $(\sqrt{\Phi})^n$, $(\sqrt{\Phi})^{n+1}$ and $(\sqrt{\Phi})^{n+2}$ respectively (Φ is the golden mean). It is easy to check that each $\mathcal{K}_n$ is a Kepler triangle.

Following Pennisi (see [8]), we will say that a triangle (not necessarily right) is *continue* if the lengths of whose sides, $a > b > c$ satisfy the proportion: $a{:}b = b{:}c$. In [8] Pennisi studied the connection between Kepler triangles and spirals, and