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A new efficient method for cases of the singular integral equation of the first kind

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ABSTRACT

Various cases of Cauchy type singular integral equation of the first kind occur rather frequently in mathematical physics and possess very unusual properties. These equations are usually difficult to solve analytically, and it is required to obtain approximate solutions. This paper investigates the numerical solution of various cases of Cauchy type singular integral equations using reproducing kernel Hilbert space (RKHS) method. The solution $u(x)$ is represented in the form of a series in the reproducing kernel space, afterwards the n-term approximate solution $u_n(x)$ is obtained and it is proved to converge to the exact solution $u(x)$. The major advantage of the method is that it can produce good globally smooth approximate solutions. Moreover, in this paper, an efficient error estimation of the RKHS method is introduced. Finally, numerical experiments show that our reproducing kernel method is efficient.

1. Introduction

The Cauchy type singular integral equation was created early in the 20th century, which has undergone an intense growth during the last years [1,2]. This type of equations has a wide applications in many branches of science and engineering like aerodynamics [3] and fracture mechanics [4]. A general form of Cauchy Singular integral equations is given as [5]:

$$\int_{-1}^{1} \frac{u(t)}{t-x}\,\mathrm{d}t + \int_{-1}^{1} k(x,t)u(t)\,\mathrm{d}t = f(x), \quad -1 < x < 1, \tag{1.1}$$

where $f(x)$ is a given function and $u(t)$ is an unknown function. When $k(x,t) = 0$ in Eq. (1.1), it is diminished to the following airfoil equation in aerodynamics

$$\int_{-1}^{1} \frac{u(t)}{t-x}\,\mathrm{d}t = f(x), \quad -1 < x < 1. \tag{1.2}$$

In [6] for four different cases, the complete analytical solution of (1.2) is explained. Let it be denoted by

$$u(x) = u_i(x), \tag{1.3}$$

where $i = 1, 2, 3, 4$ exhibits the case A, case B, case C, case D, respectively.

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Case A. The solution is unbounded at both the end points $x = \pm 1$,

$$u_1(x) = -\frac{1}{\pi^2\sqrt{1-x^2}}\int_{-1}^{1}\frac{\sqrt{1-t^2}}{t-x}f(t)\,\mathrm{d}t + \frac{P}{\sqrt{1-x^2}}, \tag{1.4}$$

where

$$P = \int_{-1}^{1} u_1(t)\,\mathrm{d}t.$$

Case B. The solution is bounded at both the end points $x = \pm 1$,

$$u_2(x) = -\frac{\sqrt{1-x^2}}{\pi^2}\int_{-1}^{1}\frac{f(t)}{(t-x)\sqrt{1-t^2}}\,\mathrm{d}t, \tag{1.5}$$

provided that

$$\int_{-1}^{1}\frac{f(t)}{\sqrt{1-t^2}}\,\mathrm{d}t = 0.$$

Case C. The solution is bounded at the end point $x = -1$, but unbounded at $x = 1$

$$u_3(x) = -\frac{1}{\pi^2}\sqrt{\frac{1-x}{1+x}}\int_{-1}^{1}\sqrt{\frac{1+t}{1-t}}\frac{f(t)}{(t-x)}\,\mathrm{d}t. \tag{1.6}$$

Case D. The solution is bounded at the end point $x = 1$, but unbounded at $x = -1$

$$u_4(x) = -\frac{1}{\pi^2}\sqrt{\frac{1+x}{1-x}}\int_{-1}^{1}\sqrt{\frac{1-t}{1+t}}\frac{f(t)}{(t-x)}\,\mathrm{d}t. \tag{1.7}$$

A variety of good numerical techniques for solving Cauchy singular integral equations are accessible in writings but search for better is always there. So many different methods have been developed to obtain an approximate solution of a Cauchy integral equation such as iteration method [7], Bernstein polynomials method [8], Jacobi polynomials method [9], Cubic spline method [10], rational functions method [11], generalized inverses method [12] and a polynomial expansion for the unknown by Mohankumar and Natarajan [13]. Bonis and Laurita [14] have proposed a Nyström method to approximate the solutions of Cauchy singular integral equations with constant coefficients having a negative index. Recently Setia [15] investigated a numerical method for approximate solution of Cauchy type singular integral equation of the first kind over $[-1, 1]$ and it is based on Bernstein polynomial and so on [16].

In recent years the reproducing kernel methods have been used to solve singular integral equations types, linear integro-differential difference equations and the kind of singular integral equations with difference kernel [17–22]. This method can be applied to solve singular integral equations since it is well known that singular integral operators on L^2 space are bounded linear operators, see [23]. In this paper, we outline a reliable strategy by reproducing kernel method for solving Cauchy type singular integral equation of the first kind.

A brief outline of this paper is as follows: A RKHS $W_2^m[-1, 1]$ is introduced in Section 2. The representation of exact solution for various cases of Cauchy type singular integral equations are obtained in Section 3. The error estimation is presented in Section 4. Numerical examples are provided in Section 5. Section 6 ends this paper with a brief conclusion.

2. Preliminaries

In the section, a RKHS $W_2^m[-1, 1]$ is introduced for solving Eq. (1.2). The representation of reproducing kernel becomes simple by improving the definition of traditional inner product (see [24]) in $W_2^m[-1, 1]$.

Definition 2.1 (*Reproducing Kernel*)**.** Let X be a nonempty abstract set. A function $R : X \times X \to \mathbb{C}$ is a reproducing kernel of the Hilbert space H if and only if

1. $\forall x \in X, R_y(.) \in H$,
2. $\forall x \in X, \forall u \in H, \big(u, R_y(.)\big) = u(y)$.

The last condition is called the reproducing property and the value of the function u at the point x is reproduced by the inner product of u with $R_y(.)$.