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On Mittag-Leffler distributions and related stochastic processes

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ABSTRACT

Random variables with Mittag-Leffler distribution can take values either in the set of non-negative integers or in the positive real line. They can be of two different types, one (type-1) heavy-tailed with index $\alpha \in (0, 1)$, the other (type-2) possessing all its moments. We investigate various stochastic processes where they play a key role, among which: the discrete space/time Neveu branching process, the discrete-space continuous-time Neveu branching process, the continuous space/time Neveu branching process (CSBP) and renewal processes with rare events. Its relation to (discrete or continuous) self-decomposability and branching processes with immigration is emphasized. Special attention will be paid to the Neveu CSBP for its connection with the Bolthausen–Sznitman coalescent. In this context, and following a recent work of Möhle (2015), a type-2 Mittag-Leffler process turns out to be the Siegmund dual to Neveu's CSBP block-counting process arising in sampling from $PD\left(e^{-t}, 0\right)$. Further combinatorial developments of this model are investigated.

1. Sibuya random variables (rvs) and related branching processes

We first investigate a class of integral-valued rvs that will show important for our general purpose.

1.1. Sibuya rvs and related ones

We start with their definition and main properties.

• *One parameter Sibuya* (α) *rv*. Let $X_\alpha \geq 1$ be an integer-valued random variable with support $\mathbb{N} = \{1, 2, \ldots\}$ defined as follows:

$$X_\alpha = \inf\left(l \geq 1 : \mathcal{B}_\alpha(l) = 1\right),$$

where $\left(\mathcal{B}_\alpha(l)\right)_{l\geq 1}$ is a sequence of independent Bernoulli rvs obeying $\mathbf{P}\left(\mathcal{B}_\alpha(l) = 1\right) = \alpha/l$ where $\alpha \in (0, 1)$. It is thus the first epoch of a success in a Bernoulli trial when the probability of success is inversely proportional to the number of the trial. X_α is called a Sibuya(α) rv. Then

$$\mathbf{P}\left(X_\alpha = k\right) = (-1)^{k-1}\binom{\alpha}{k}, \quad k \geq 1,$$

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with $\binom{\alpha}{k} = (\alpha)_k/k!, (\alpha)_k := \Gamma(\alpha+1)/\Gamma(\alpha+1-k) = \alpha(\alpha-1)\cdots(\alpha-k+1)$, the Pochhammer's symbol (or decreasing factorial). Its probability generating function (pgf) is

$$\phi_\alpha(z) := \mathbf{E}\left(z^{X_\alpha}\right) = 1-(1-z)^\alpha, \quad z \leq 1.$$

We note that $\mathbf{P}(X_\alpha = k)$ is also $\mathbf{P}(X_\alpha = k) = \alpha[\overline{\alpha}]_{k-1}/k!$, where $\overline{\alpha} := 1-\alpha$ and $[a]_k := a(a+1)\cdots(a+k-1), k \geq 1$, are the rising factorials of a with $[a]_0 := 1$.

• *Discrete-stable* (μ, α) *rv* [1]. Consider the random variable $S_{\mu,\alpha}$ given by the random sum

$$S_{\mu,\alpha} = \sum_{l=0}^{P_\mu} X_\alpha(l),$$

where P_μ is Poisson distributed with mean $\mu > 0$ and $(X_\alpha(l))_{l\geq 0}$ is an iid sequence of Sibuya(α) rvs ($X_\alpha(l) \stackrel{d}{=} X_\alpha$), independent of P_μ. Then $\phi_{P_\mu}(z) = \mathbf{E}\left(z^{P_\mu}\right) = e^{-\mu(1-z)}$ and

$$\phi_{S_{\alpha,\mu}}(z) = \phi_{P_\mu}(\phi_\alpha(z)) = e^{-\mu(1-z)^\alpha}$$

the pgf of a discrete-stable(α, μ) rv, say $S_{\alpha,\mu}$. We will come back to this distribution below. Note that, with $S_\alpha := S_{\alpha,1}$, and in view of $S_{\alpha,\mu} \stackrel{d}{=} \mu^{1/\alpha} \circ S_\alpha$, μ is the scale parameter of $S_{\alpha,\mu}$.

• *Scaled Sibuya* (α, λ) *rv*. Let $c \in (0, 1)$. Define the c-thinned version of the rv X_α, say $X_{\alpha,c} := c \circ X_\alpha$, as the random sum

$$X_{\alpha,c} = c \circ X_\alpha \stackrel{d}{=} \sum_{l=1}^{X_\alpha} B_c(l),$$

with $(B_c(l))_{l\geq 1}$ a sequence of independent and identically distributed (iid) Bernoulli variables such that $\mathbf{P}(B_c(1) = 1) = c$, independent of X_α. This binomial thinning operator, acting on discrete rvs, has been defined by [1]; it stands as the discrete version of the change of scale (note that if $X = n$ is a constant integral rv, $c \circ X$ is random with bin(n, c) distribution). The pgf of $X_{\alpha,c}$ is

$$\phi_{\alpha,c}(z) := \mathbf{E}\left(z^{X_{\alpha,c}}\right) = \phi_{X_\alpha}(1-c(1-z)) = 1-(c(1-z))^\alpha, \quad z \leq 1.$$

With $\lambda = c^\alpha \in (0, 1)$, we shall therefore call a rv $X_{\alpha,\lambda}$ with pgf $\phi_{\alpha,\lambda}(z) = 1-\lambda(1-z)^\alpha$ a scaled Sibuya(α, λ) rv, with scale parameter λ, obeying $X_{\alpha,\lambda} \stackrel{d}{=} \lambda^{1/\alpha} \circ X_\alpha$. $X_{\alpha,\lambda} \geq 0$ is now an integer-valued random variable with support $\mathbb{N}_0 = \{0, 1, 2, \ldots.\}$, satisfying

$$\pi_{\alpha,\lambda}(0) := \mathbf{P}\left(X_{\alpha,\lambda} = 0\right) = 1-\lambda \quad \text{and}$$

$$\pi_{\alpha,\lambda}(k) := \mathbf{P}\left(X_{\alpha,\lambda} = k\right) = \lambda(-1)^{k-1}\binom{\alpha}{k} = \alpha\lambda\frac{[\overline{\alpha}]_{k-1}}{k!}, \quad k \geq 1. \tag{1}$$

Both $X_{\alpha,\lambda}$ and $X_\alpha = X_{\alpha,1}$ are heavy-tailed with exponent α: $\mathbf{P}(X > k) = L(k)k^{-\alpha}$ for some slowly-varying sequence $L(k)$.

• *Main properties* [2]. The rv $X_{\alpha,\lambda}$ is infinitely divisible (ID), or compound Poisson, iff $\lambda \leq 1-\alpha$. This follows from the fact that, with $\mu = -\log(1-\lambda) \leq -\log\alpha$

$$\phi_{\alpha,\lambda}(z) = 1-\lambda(1-z)^\alpha = e^{-\mu(1-h(z))}$$

for some absolutely monotone pgf $h(z)$ (the pgf of the sizes of the batches), obeying $h(0) = 0$.

It is even discrete self-decomposable (and thus unimodal) iff $\lambda \leq (1-\alpha)/(1+\alpha)$ with $X_{\alpha,\lambda}$ self-decomposable $\Rightarrow X_{\alpha,\lambda}$ ID, [1]. We will come back to this self-decomposability property below.

• *Three-parameters Sibuya* (α, β, λ) *rv*. Let $\beta > 0$. If $X_{\alpha,\lambda}$ is ID (else if $\lambda \leq 1-\alpha$), then for all $\beta > 0$

$$\phi_{\alpha,\beta,\lambda}(z) = (1-\lambda(1-z)^\alpha)^\beta$$

is the pgf of some rv $X_{\alpha,\beta,\lambda}$, called a generalized Sibuya(α, β, λ) rv. This is because, under our assumptions, $X_{\alpha,\lambda}$ is compound Poisson.

1.2. Branching processes involving Sibuya rvs: discrete space–time Neveu process

We describe here an integral-valued Bienaymé–Galton–Watson branching process in discrete time whose branching mechanism is a Sibuya(α, λ) rv. It turns out that the population size at generation n is itself again a Sibuya(α_n, λ_n) rv, so computable. We call it the discrete Neveu process. We investigate some of the consequences of this remarkable fact.

• *Branching process with Sibuya* (α, λ) *offspring distribution (discrete-time)*. Let $\phi_{\alpha_1,\lambda_1}(z)$ and $\phi_{\alpha_2,\lambda_2}(z)$ be the pgfs of two independent scaled Sibuya rvs with parameters (α_1, λ_1) and (α_2, λ_2). We have the stability under composition property

$$\phi_{\alpha_2,\lambda_2}\left(\phi_{\alpha_1,\lambda_1}(z)\right) = \phi_{\alpha_2\alpha_1,\lambda_2\lambda_1^{\alpha_2}}(z).$$