Journal of Computational and Applied Mathematics 296 (2016) 212–227

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

On control polygons of Pythagorean hodograph septic curves ☆

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ARTICLE INFO

Article history:
Received 1 August 2014
Received in revised form 2 September 2015

Keywords:
Septic
Pythagorean hodograph
Bézier curves
Control polygon
Compatibility
Transition curves

ABSTRACT

For a given septic Bézier curve with a distinct ordered sequence of control points, how to determine whether it is a PH curve via exact symbolic computation in theory. This problem motivated the study of a necessary and sufficient condition for a planar septic Bézier curve to possess a Pythagorean hodograph (PH). Based on the definition of a PH curve and the complex representation of a planar curve, we develop geometric conditions in terms of the leg-lengths and angles of a control polygon that must be separated to guarantee the PH property. The relation between the compatibility of solutions with respect to the complex coefficients of PH equations and geometric constraints is analyzed. Moreover, PH septic curves with inflections are extended to construct S-shaped transition curves.

1. Introduction

Pythagorean hodograph (PH) curves are an important subclass of polynomial parametric curves. Their most remarkable properties [1,2] are that their arc-length function is piecewise polynomial, and that the fixed-distance offsets of PH curves are rational polynomial curves, which can be exactly represented in CAD systems. These properties are especially advantageous in applications such as Computer Numerical Control (CNC) machining of digital motion along curved paths, robotics [3], and the design of highways and railways. Since they were first introduced, PH curves have been exhaustively studied. The most important task in CAGD is to design fair curves that meet user requirements. Studies on PH curves encompass the construction of PH curves or PH spline curves with various continuous orders via Hermite interpolation, and these processes involve solving algebraic equations. Meek and Walton [4] found explicit formulae for PH cubic curves by geometric Hermite interpolation. Li and Deng [5] further extended the technique of G^1 Hermite interpolation by a PH cubic. PH quintic splines with C^2 continuous order have been constructed by the homotopy method [6,7], while Moon and Farouki [8] developed efficient and reliable algorithms and gave a shape analysis for interpolation by PH quintics. G^2 Hermite interpolation with PH curves of degree seven was proposed by Jüttler [9]. Moreover, Sir and Jüttler [10] converted a curve described in G-code into a C^2 continuous PH spline curve of degree 9 by interpolation, and showed that one of the solutions has approximation order six. PH curves can also be used to construct transition curves of G^2 contact, composed of a single segment, between two circles. These are used in the design of highways and railway routes, or for aesthetic applications [11–14].

Thus far, despite their algebraic properties being exhaustively studied, only a few research studies have considered the geometric properties of PH curves. It is well known that leg-lengths and angles of control polygon are intrinsic geometric parameters without depending on choice of coordinates and their data can be obtained by measurement. Thus, the discussion about the geometric properties of PH curves in terms of the leg-lengths and angles of a control polygon is important both in theoretical terms and for practical applications. However, earlier studies only formulated PH cubic and

☆ Project supported by the National Natural Science Foundation of China (No. 61272300).

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http://dx.doi.org/10.1016/j.cam.2015.09.006
0377-0427/

quintic curves as geometric constraints on a polygon possessing the PH property. In particular, Farouki and Sakkalis [2] showed that a cubic Bézier curve with control points $\{\mathbf{P}_j\}_{j=0}^3$ is a PH curve if and only if $L_1^2 = L_0L_2$ and $\theta_1 = \theta_2$, where L_j is the length of the leg $\mathbf{P}_j\mathbf{P}_{j+1}$ of the control polygon, $j = 0, 1, 2$, and θ_1, θ_2 are interior angles at $\mathbf{P}_1$ and $\mathbf{P}_2$. Although the necessary and sufficient condition for the cubic case may be expressed in terms of the leg-lengths and angles of the control polygon, it is worth noting that the conditions for leg-lengths and the conditions for angles are represented separately. Thus, for a cubic Bézier curve with distinct control points, we can know whether it is a PH curve by the above-mentioned geometric constraints, which are only related to leg-lengths and angles. Moreover, Farouki and Sakkalis [2] gave the geometric conditions for the PH quintic case, but these cannot be formulated in such a way that the leg-lengths and angles can be separated. The geometric conditions of PH curves besides cubic and quintic were not studied for a long time following their first introduction. In recent years, Wang and Fang [15] have presented a necessary and sufficient geometric characterization of a PH quartic in terms of the leg-lengths and angles of a control polygon. This contains three equations with respect to the leg-lengths and an angle condition; based on their result, a novel geometric approach to G^1 Hermite interpolation by planar PH quartics was proposed. Yong and Zheng [16] derived a geometric relationship among the Bézier control points of PH quintics, and generalized the geometric method for Hermite interpolation. Later, Fang [17] deduced the geometric characterization of planar non-primitive PH quintics in the same way as for the PH quartic curve, and also discussed the problem of C^1 Hermite interpolation by non-primitive PH quintics. Fang et al. [18] then studied the geometric properties of PH sextic curves.

Concerning the PH septic curve, Jüttler [9] proposed a scheme for G^2 Hermite interpolation by PH curves of degree seven based on certain algebraic properties. Motivated by recent work [15,18], we further present the necessary and sufficient geometric constraints for the PH septic. These are expressed in terms of separate conditions on the leg-lengths and angles of the control polygon.

2. Description of PH septic curve

Complex analysis has been demonstrated to be a powerful tool for curve design and analysis. It is well known that a plane point or vector (x, y) may be regarded as a complex number $\mathbf{z} = x + \mathbf{i}y(\mathbf{i} = \sqrt{-1})$. In this paper, let $\sqrt{\mathbf{z}^2}$ be $\mathbf{z}$. We are familiar with the polar form $\mathbf{z} = re^{\mathbf{i}\theta}$ of a complex number, where r denotes the modulus of $\mathbf{z}$ and $\theta = \arg(\bullet)$ represents the signed angle of the line from 0 to $\mathbf{z}$ with respect to the positive real axis (x-axis). This angle is restricted to the range $[0, \pi]$ anticlockwise and $[-\pi, 0]$ clockwise. A parametric plane curve in $\mathbf{r}(t) = (x(t), y(t))$ is associated with a complex function $\mathbf{r}(t) = x(t) + \mathbf{i}y(t)$. This sets up a one-to-one mapping between parametric plane curves and complex functions. The Bézier curve of degree n is given by

$$\mathbf{P}(t) = \sum_{j=0}^{n} \mathbf{P}_j \binom{n}{j} (1-t)^{n-j} t^j, \quad 0 \le t \le 1,$$

$\mathbf{P}_j, j = 0, \ldots, n$ are control points of the control polygon for $\mathbf{P}(t)$. Consider $\mathbf{P}(t) = (x(t), y(t))$ or $x(t) + \mathbf{i}y(t)$, $\mathbf{P}(t)$ is said to be a PH curve if $x'(t)^2 + y'(t)^2$ can be expressed as the square of a polynomial in t. To ensure that $\mathbf{P}(t)$ is indeed a PH curve, define $x'(t)$ and $y'(t)$ as [2]

$$x'(t) = u^2(t) - v^2(t), \qquad y'(t) = 2u(t)v(t)$$

where $u(t)$ and $v(t)$ are polynomials. Writing $\boldsymbol{\omega}(t) = u(t) + \mathbf{i}v(t)$, $\boldsymbol{\omega}(t)$ is defined as pre-image [2], the derivative of $\mathbf{P}(t)$ is expressed by

$$\mathbf{P}'(t) = x'(t) + y'(t)\mathbf{i} = \boldsymbol{\omega}^2(t) = u^2(t) - v^2(t) + 2u(t)v(t)\mathbf{i}. \tag{1}$$

A PH cubic curve is obtained when $n = 3$, and a PH quintic curve is obtained when $n = 5$. A PH septic curve

$$\mathbf{P}(t) = \sum_{j=0}^{7} \mathbf{P}_j \binom{7}{j} (1-t)^{7-j} t^j, \tag{2}$$

when $n = 7$ is obtained by defining pre-image $\boldsymbol{\omega}(t)$ as

$$\boldsymbol{\omega}(t) = \boldsymbol{\omega}_0(1-t)^3 + 3\boldsymbol{\omega}_1(1-t)^2t + 3\boldsymbol{\omega}_2(1-t)t^2 + \boldsymbol{\omega}_3t^3, \tag{3}$$

$\boldsymbol{\omega}_0, \boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_3$ are complex coefficients, thus $u(t), v(t)$ in (1) as

$$u(t) = u_0(1-t)^3 + 3u_1(1-t)^2t + 3u_2(1-t)t^2 + u_3t^3, \tag{4}$$

$$v(t) = v_0(1-t)^3 + 3v_1(1-t)^2t + 3v_2(1-t)t^2 + v_3t^3, \tag{5}$$

$u_0, u_1, u_2, u_3, v_0, v_1, v_2, v_3$ are real coefficients. In fact, it is true that $\boldsymbol{\omega}_j = u_j + \mathbf{i}v_j, j = 0 \ldots 3$. Let $\triangle$ be the difference operator and $\triangle\mathbf{P}_j = \mathbf{P}_{j+1} - \mathbf{P}_j = \mathbf{P}_j\mathbf{P}_{j+1}, j = 0, \ldots, 6$ be the seven directed legs of the control polygon. The derivative of PH septic written by (2) is

$$\mathbf{P}'(t) = 7\sum_{j=0}^{6} \triangle\mathbf{P}_j \binom{6}{j} (1-t)^{6-j} t^6. \tag{6}$$