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New simultaneous root-finding methods with accelerated convergence for analytic functions

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ABSTRACT

A new iterative method of the fourth order for the simultaneous determination of zeros of a class of analytic functions, is proposed. Further improvements of the basic method are attained by using Newton's and Halley's corrections giving the orders of convergence five and six, respectively. The improved convergence is achieved with negligible number of additional calculations, which significantly increases the computational efficiency of the accelerated methods. Numerical examples demonstrate a good convergence properties, fitting very well theoretical results.

1. Introduction

The goal of this paper is to present the construction and convergence analysis of a new simultaneous method for the computation of real or complex zeros of a wide class of analytic functions. Considered analytic functions from this class have only simple zeros inside a simple smooth closed contour in the complex plane. In Section 2 we derive a new method. The derivation is based on a new *zero-relation* for analytic functions. Convergence analysis presented in Section 3 shows that the order of convergence of the considered method is four. In the same section we state accelerated methods using Newton's and Halley's corrections. Faster convergence of accelerated methods is attained with only few additional operations, which provides a high computational efficiency of these methods. Numerical examples, given in Section 4, demonstrate the convergence behavior of the considered methods and confirm theoretical results.

2. Derivation of a fixed point relation

Let $z \mapsto \Phi(z)$ be an analytic function inside and on the simple smooth closed contour Γ, without zeros on Γ and with a known number n of simple zeros inside Γ. Then, following Smirnov [14], Φ can be expressed in the form of product

$$\Phi(z) = \Psi(z)\prod_{j=1}^{n}(z-\zeta_j)$$

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inside Γ. Here $\zeta_1, \ldots, \zeta_n$ are the zeros of Φ (inside Γ) and Ψ is an analytic function such that $\Psi(z) \neq 0$ for all $z \in \operatorname{int} \Gamma$, where the symbol int Γ means interior of the curve Γ. The set of analytic functions with the described properties will be denoted by Ω.

According to Iokimidis and Anastasselou [6] the analytic function Ψ can be represented in the form

$$\Psi(z) = \exp\big(Y(z)\big)$$

inside Γ, where Y is also an analytic function inside Γ given by

$$Y(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\log\left[(w-\eta)^{-n}\Phi(w)\right]}{w-z}\,dw \tag{1}$$

and η is an arbitrary point inside Γ. Hence,

$$\Phi(z) = \exp\big(Y(z)\big)\prod_{j=1}^{n}(z-\zeta_j). \tag{2}$$

From (1) we find

$$Y'(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\log\left[(w-\eta)^{-n}\Phi(w)\right]}{(w-z)^2}\,dw, \tag{3}$$

where from

$$Y''(z) = \frac{1}{\pi i}\int_\Gamma \frac{\log\left[(w-\eta)^{-n}\Phi(w)\right]}{(w-z)^3}\,dw.$$

Applying integration by parts, from (3) we obtain

$$Y'(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\Phi'(w)}{\Phi(w)}\,\frac{dw}{w-z} \tag{4}$$

(see Smirnov [14]). Then the second derivative is given by

$$Y''(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\Phi'(w)}{\Phi(w)}\,\frac{dw}{(w-z)^2}. \tag{5}$$

As well known, the number of zeros n of Φ inside Γ may be determined by the *argument principle* (see, e.g., Henrici [5])

$$n = \frac{1}{2\pi i}\int_\Gamma \frac{\Phi'(w)}{\Phi(w)}\,dw = n\big(\Phi(\Gamma), 0\big). \tag{6}$$

$\Phi(\Gamma)$ denotes the image of the curve Γ under the mapping Φ. The integer $n\big(\Phi(\Gamma), 0\big)$ is the so-called *winding number* of $\Phi(\Gamma)$ with respect to the origin and it is equal to the number of times that the curve $\Phi(\Gamma)$ "winds" itself around the origin.

In what follows we will derive a zero-relation convenient for the construction of simultaneous approximation of zeros of a given analytic function from the set Ω. For distinct sufficiently close approximations $z_1, \ldots, z_n$ to the zeros $\zeta_1, \ldots, \zeta_n$ of the $\Phi(z)$ we introduce the notations:

$$u(z) = \frac{\Phi(z)}{\Phi'(z)}, \qquad u_i = u(z_i), \qquad \varepsilon_i = z_i - \zeta_i, \qquad Y_i = Y(z_i) \quad i \in I_n := \{1, \ldots, n\},$$

$$A_k(z) = \Phi^{(k)}(z)/(k!\Phi'(z)), \qquad A_{k,i} = A_k(z_i), \quad k = 2, 3, \ldots,$$

$$\Sigma_{q,i}(z) = \sum_{j\in I_n\setminus\{i\}} \frac{1}{(z-\zeta_j)^q}, \qquad \Sigma_{q,i} = \Sigma_{q,i}(z_i), \quad q = 1, 2.$$

We will write $w = O_M(z)$ if two real or complex numbers w and z have moduli of the same order, that is, $|w| = O(|z|)(\varepsilon \to 0)$, where $\varepsilon = \max_{1\le i\le n}|z_i - \zeta_i|$.

Using the introduced notations we obtain

$$|\Sigma_{q,i}| = |\Sigma_{q,i}(z_i)| \le \sum_{j\in I_n\setminus\{i\}} \frac{1}{|z_i-\zeta_j|^q} \le \frac{n-1}{d^q}, \quad q = 1, 2, \tag{7}$$

where

$$d = \min_{1\le i\le n} d_i, \qquad d_i = \min_{\substack{1\le j\le n\\ j\neq i}} |z_i - \zeta_j| > 0.$$