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Numerical analysis and simulation for a nonlinear wave equation

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ABSTRACT

In this work we study a nonlinear wave equation, depending on different norms of the initial conditions, has bounded solution for all $t > 0$ or $0 < t < T_0$ for some $T_0 > 0$. We also prove that the solution may blow-up at T_0. Proofs of some the analytical results listed are sketched or given. For approximate numerical solutions we use the finite element method in the spatial variable and the finite difference method in time. The nonlinear system for each time step is solved by Newton's modified method. We present numerical analysis for error estimates and numerical simulations to illustrate the convergence of the theoretical results. We present too, the singularity points $(x^*, \ t^*)$, where the blow-up occurs for different ρ values in a numerical simulation.

1. Introduction

Let Ω be a bounded open set of $\mathbb{R}^n$, with C^1boundary and Q be the cylinder $\Omega \times (0, T)$ of $\mathbb{R}^{n+1}$ for $T > 0$, with lateral boundary represented by $\Sigma = \Gamma \times (0, T)$. We shall consider the following nonlinear problem:

$$\begin{cases} u_{tt} - \Delta u + |u|^\rho = f \quad \text{in } Q, \quad \rho > 1 \\ u = 0 \quad \text{on } \Sigma, \\ u(0, x) = u_0(x), \qquad u_t(0, x) = u_1(x), \quad \forall x \in \Omega. \end{cases} \tag{1}$$

The solution of problem (1) is a function $u = u(x, t)$ depending on time t and spatial variable x.

A special case of this problem, with $\rho = 2$, has been studied in [1], and despite the simple form, the problem (1) has interesting properties. To illustrate the difficulties involved, we shall follow the classical procedure, to obtain the energy inequality for the wave equation and consider $f = 0$ for simplicity.

Applying the inner product in $L^2(\Omega)$ of Eq. $(1)_1$ with u_t and integrating over $[0, \ t]$, provided that there has been sufficient regularity of terms of the partial differential equations (1), we obtain the following inequality

$$|u_t(t)|^2 + |\nabla u(t)|^2 + \frac{2}{\rho+1}\int_\Omega u(t)|u(t)|^\rho d\Omega \le |u_1|^2 + |\nabla u_0|^2 + \frac{2}{\rho+1}\int_\Omega u_0|u_0|^\rho d\Omega, \tag{2}$$

where, $|\cdot|$ is the norm in $L^2(\Omega)$.

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The term involving the integral, has no definite sign, so the relation (2) cannot ensure that $|u(t)|$ and $|\nabla u(t)|$ are limited. In fact, it has been proved, (see [1] for $\rho = 2$ or [2]), that a solution of (1) exists for sufficiently small initial data and source term, for every $T > 0$. In addition, it was proved in [2] that for any initial conditions, the solution is restricted to some time interval and we will confirm this result theoretically and numerically.

Furthermore, in [3], it was proved for a more general nonlinear partial differential equations, that for sufficiently large initial conditions even without the source term, the solution blows up in finite time. The hypotheses in that work are not satisfied by the solution of the problem (1), especially the requirement that it be of class C^2. But an adaptation of the technique leads to similar result displaying the blowing up, using conditions compatible with the existence theorem.

On the other hand, in numerical analysis, we are primarily interested in semi-discrete error and fully discrete error analysis and numerical simulation, rather than aspects of energy conservation of the wave equation. There are many numerical results concerning energy behavior under the presence of a dissipative mechanism for hyperbolic equation, among them we can mention [4,5].

In [6], they investigated the question of finding a numerical schemes that preserve rigorously the discrete energy, for nonlinear wave equations. Such schemes are well known in the linear case, and conservation of energy automatically provides the stability of the Newmark scheme when $\theta \geq 1/4$ since the energy is always positive. In particular, when $\theta = 0$, then we have the well-known leap-frog scheme, for which a discrete energy is also conserved.

There are a variety of numerical methods using finite element discretizations for hyperbolic equation of second order, among them we can mention, Dupont [7] and Wheeler [8], who obtained error estimates for the approximate solution of hyperbolic equation using the Galerkin method.

An outline of this paper follows: Section 2, we will only state the existence and uniqueness theorem of solution to the problem (1). If restrictions on norms of u_0, u_1 and f are satisfied the solution applies for all $t > 0$. Moreover, the solution is local in t if these are not met. Furthermore, in this case, it ensures that the standard $L^p(\Omega)$ norm of $u(x,t)$ will tend to infinity as t tends to a given t_0, for all $p \in [1, \infty]$ and we shall prove this result.

In Section 3, we present the numerical method from the variational formulation, with the finite element method for spatial discretization and the finite difference method for solving a system of differential equations of second order in time. It also presented the modified Newton's method for solving nonlinear system in one dimensional case.

In Section 4, we present some numerical simulations to validate the numerical method employed. Some tables for various meshes showing the order of convergence as well as the graphs of numerical solutions are also presented and tables with singularity points (x^*, t^*) where the blow-up occurs, for different ρ values.

2. Analytical results

Let $((\cdot,\cdot))$, $\|\cdot\|$ and $(\cdot,\cdot)$, $|\cdot|$ be respectively the scalar product and the norms in $H_0^1(\Omega)$ and $L^2(\Omega)$. Thus, when we write $|u| = |u(t)|$, $\|u\| = \|u(t)\|$ it will mean the $L^2(\Omega)$, $H_0^1(\Omega)$ norm of $u(x,t)$ respectively. For convenience, we will use the prime ($'$) to denote the derivative with respect to time t.

For the next theorem, we need that the initial data are small, i.e., consider the following hypothesis:

(H1) $\|u_0\| < C_0^{-\left(\frac{\rho+1}{\rho-1}\right)}$, $\gamma(u_0, u_1) < \left(\frac{\rho-1}{\rho+1}\right) C_0^{-2\left(\frac{\rho+1}{\rho-1}\right)}$,

where C_0, is the constant immersion of $H_0^1(\Omega)$ in $L^{\rho+1}(\Omega)$ and $\gamma(u_0,u_1)$ as defined in the following theorem.

The proof of the following theorem, can be found in [2].

Theorem 1. *Let Ω be an open set, limited to $\mathbb{R}^n$ with regular boundary and $\rho > 1$ satisfying $\rho \leq \frac{2n}{n-2}$, if $n \geq 3$. Consider $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$, $f \in L^1(0,\infty; L^2(\Omega))$, and the function*

$$\gamma(u_0,u_1) = \left(|u_0|^2 + \|u_1\|^2 + \frac{2}{\rho+1}\int_\Omega |u_0|^\rho u_0 d\Omega + \|f\|_{1,2}\right) \exp^{\|f\|_{1,2}}, \tag{3}$$

where $\|\cdot\|_{1,2}$ is the norm in $L^1(0,\infty; L^2(\Omega))$. Suppose that the initial data satisfy the hypothesis (H1). *Then, there is an unique non local solution to the problem (1), for all $T > 0$, satisfying $u \in L^\infty(0,T; H_0^1(\Omega))$ and $u_t \in L^\infty(0,T; L^2(\Omega))$.*

On the other hand, if the hypothesis (H1) *is not satisfied, there is T_0, dependent of $\{\rho, \|u_o\|, |u_1|, f\}$, such that the problem (1) has a local solution satisfying $u \in L^\infty(0,T_0; H_0^1(\Omega))$ and $u_t \in L^\infty(0,T_0; L^2(\Omega))$.*

Proof. In this proof, we will do the same procedure as the demonstrations in [2] with some changes that simplify the calculus. Due to restrictions in the available space, the more traditional techniques were omitted and can be found in [1] or [2].

Let $(w_n)_{n\in\mathbb{N}}$ be an orthonormal system in $L^2(\Omega)$, whose space generated is contained and dense in $H_0^1(\Omega)$. We represent by V_m the subspace generated by vectors $\{w_1, w_2, \ldots, w_m\}$. We propose the following approximate problem: Determine $u_m : [0, T_m) \mapsto V_m$, so that:

$$\begin{cases} (u_m''(t), w) + (\nabla u_m(t), \nabla w) + (|u_m(t)|^\rho, w) = (f(t), w), \quad \forall w \in V_m \\ (u_m(0), w) = (u_0, w), \\ (u_m'(0), w) = (u_1, w). \end{cases} \tag{4}$$