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First passage times of two-dimensional correlated processes: Analytical results for the Wiener process and a numerical method for diffusion processes

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ABSTRACT

Given a two-dimensional correlated diffusion process, we determine the joint density of the first passage times of the process to some constant boundaries. This quantity depends on the joint density of the first passage time of the first crossing component and of the position of the second crossing component before its crossing time. First we show that these densities are solutions of a system of Volterra–Fredholm first kind integral equations. Then we propose a numerical algorithm to solve it and we describe how to use the algorithm to approximate the joint density of the first passage times. The convergence of the method is theoretically proved for bivariate diffusion processes. We derive explicit expressions for these and other quantities of interest in the case of a bivariate Wiener process, correcting previous misprints appearing in the literature. Finally we illustrate the application of the method through a set of examples.

1. Introduction and motivation

The first passage time (FPT) problem of univariate stochastic processes through boundaries is relevant in different fields, e.g. economics [1], engineering [2], finance [3,4], neuroscience [5,6], physics [7], psychology [8] and reliability theory [9,10]. For one-dimensional processes, the FPT problem has been widely analytically investigated both for constant and time dependent boundaries [11,12,5], yielding explicit expressions for the FPT density of the Wiener process [13], of a special case of the Ornstein–Uhlenbeck (OU) process [14], of the Cox–Ingersoll–Ross (also known as Feller or square-root) process [15,16] and of some processes which can be obtained through suitable measure or space–time transformations of the previous processes [11,15,17]. For most of the processes arising from applications, closed form expressions are not available but it was proved that the FPT distribution function is solution of integral equations. This has determined the development of ad hoc numerical methods for the solution of Volterra integral equations of the first and second types arising from both the direct and the inverse FPT problem [18–23].

Results for the FPT problem of bivariate processes are still scarce and fragmentary. Analytic results are available for bivariate FPTs through specific surfaces [24,25], for the FPTs of a Wiener and of an integrated process [26–29] and for the FPTs of two correlated Wiener processes with zero [30–32] or positive drift [33] in presence of absorbing boundaries.

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The main goal of this paper is to investigate the bivariate joint distribution of the hitting times of a bivariate diffusion process. A new difficulty arises with respect to the univariate case: the dynamics of the process after the first crossing depend on the type of considered boundaries. Indeed the first component attaining its boundary can stop its evolution, be absorbed there or pursue its evolution depending on whether the boundaries are *killing*, *absorbing* or *crossing*, respectively. In all cases, the slowest component evolves till its passage time. The different boundary conditions are defined in Section 2 together with some further mathematical background. The different scenarios are studied in Section 3, where we also derive the joint FPT densities in the three cases. Conscious of the important role of Volterra Integral equations in the univariate FPT problem, here we extend the approach used in the one dimensional case [18] or in the FPT problem of a component of a Gauss Markov process [34]. These quantities depend on the joint densities of the second crossing component before its FPT and the FPT of the first crossing component, which we show to be the solutions of a system of Volterra–Fredholm first kind integral equations [35]. In Section 4 we propose a numerical method to solve the system and we describe how to obtain the joint FPT density using our algorithm. Since the dynamics of the process before the first crossing time are the same for all types of boundaries, the proposed method can always be used. In Section 5 we prove the convergence of the algorithm and we study its order of convergence. A useful feature of the proposed algorithm is that it allows to avoid the prohibitive computational effort required for simulating the joint density of the FPTs [36]. Indeed it allows to switch from a Monte Carlo simulation method [37] to a deterministic numerical method.

To numerically illustrate the convergence of the method, we consider two correlated Wiener processes and compare the theoretical and the numerical results. The desired joint density of the second crossing component before its FPT and the FPT of the first crossing component can be obtained starting from the joint density of the process constrained to be below the boundaries, which is available in [33,31,37]. The formulas for the driftless case presented in [31] contain misprints, which have been independently corrected in [33,37]. In [31] the case with drift is also considered, but unfortunately some expressions present further misprints. Since we have not been able to locate correct results elsewhere in the literature, in Section 6 we correct these formulas and determine other quantities of interest. In particular we calculate the joint density of the position of the process constrained to be below the boundaries, of the FPTs both with and without drift and of the second crossing component before its FPT and the FPT of the first crossing component. A comparison of this last density with its numerical approximation obtained using the algorithm is presented in Section 7. There we also illustrate the application of our method to approximate the joint FPT density of a bivariate OU process with correlated components. This is particularly relevant in neuroscience, where FPTs are used to describe neural action potentials (spikes) and multivariate OU processes can be used to model neural networks, as recently discussed in [38].

2. Mathematical background

Consider a two-dimensional time homogeneous diffusion process $\mathbf{X} = \left\{(X_1, X_2)'(t); t > t_0\right\}$, solution of the stochastic differential equation

$$d\mathbf{X}(t) = \boldsymbol{\mu}(\mathbf{X}(t))dt + \boldsymbol{\Sigma}(\boldsymbol{X}(t))\, d\mathbf{W}(t), \qquad \mathbf{X}(t_0) = \mathbf{x}_0 = \left(x_{01}, x_{02}\right)', \quad t > t_0, \tag{1}$$

where $'$ indicates vector transpose. Here $\boldsymbol{W}(t)$ is a two-dimensional standard Wiener process, the $\mathbb{R}^2$-valued function $\boldsymbol{\mu}$ and the $\mathbb{R}^2 \times \mathbb{R}^2$ matrix-valued function $\boldsymbol{\Sigma}$ are assumed to be defined and measurable on $\mathbb{R}^2$ and all the conditions on existence and uniqueness of the solution are satisfied [39].

Define the random variable

$$T_i = \inf\{t > t_0 : X_i(t) > B_i\} \quad i = 1, 2,$$

i.e. the FPT of X_i through the constant boundary $B_i > x_{0i}$. We denote by $T = \min(T_1, T_2)$ the random variable corresponding to the first exit time of $\mathbf{X}$ from the strip $(-\infty, B_1) \times (-\infty, B_2)$. Our goal is to determine the joint probability density function (pdf) of (T_1, T_2) for a process $\mathbf{X}$ originated in $\mathbf{y} = (y_1, y_2)$ at time s, defined by

$$f_{(T_1,T_2)}(t_1, t_2|\boldsymbol{y}, s) := \frac{\partial^2}{\partial t_1 \partial t_2}\mathbb{P}(T_1 < t_1, T_2 < t_2|\boldsymbol{X}(s) = \boldsymbol{y}).$$

Throughout the paper we consider the following densities for $i, j = 1, 2,\ i \neq j$ and $s < t$:

- joint pdf of the components of the process $\mathbf{X}$ up to time T, defined by

$$f^a_{\mathbf{X}}(\mathbf{x}, t|\mathbf{y}, s) := \frac{\partial^2}{\partial x_1 \partial x_2}\mathbb{P}\left(\mathbf{X}(t) < \mathbf{x}, T > t|\mathbf{X}(s) = \mathbf{y}\right);$$

- conditional pdf of X_i given X_j up to time T, defined by

$$f^a_{X_i|X_j}(x_i, t|x_j, t; \mathbf{y}, s) := \frac{\partial}{\partial x_i}\mathbb{P}(X_i(t) < x_i, T_i > t|X_j(t) = x_j, T_j > t, \mathbf{X}(s) = \mathbf{y});$$