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An iterative algorithm for G^2 multiwise merging of Bézier curves

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ABSTRACT

This paper presents an iterative algorithm for G^2 multiwise merging of Bézier curves. By using the G^2 constraint, the L_2 distance is represented after simplification as a quartic polynomial in two parameters relating to the magnitudes of end tangents of the merged curve. These two parameters are restricted in a feasible region, in order for the merged curve to preserve the specified directions of end tangents. Then G^2 multiwise merging is formulated as a constrained minimization problem, and the classic projected Newton method is applied to find the minimizer. Some extensions of multiwise merging using G^3 constraints, other energy functionals and curve representations are also outlined. Several comparative examples are provided to demonstrate the effectiveness of the proposed method.

1. Introduction

Assume that the given are $K(\geq 2)$ segments of adjacent Bézier curves of degrees n_k, $k = 1, \ldots, K$, each of which is defined in terms of the control points $\mathbf{P}_{n_k}^k := (\mathbf{p}_0^k, \ldots, \mathbf{p}_{n_k}^k)^T$ as

$$\mathbf{P}^k(t) = \sum_{i=0}^{n_k} B_i^{n_k}(t)\mathbf{p}_i^k = \mathbf{B}_{n_k}\mathbf{P}_{n_k}^k, \quad t \in [0, 1]. \tag{1}$$

Throughout, the notation $\mathbf{B}_n := \left(B_0^n(t), \ldots, B_n^n(t)\right)$ is used to denote the row vector of the Bernstein basis of degree n, with $B_i^n(t) = \binom{n}{i}t^i(1-t)^{n-i}$. The problem of multiwise merging [1,2] aims to find a single degree-n Bézier curve (called *merged curve*) with control points $\mathbf{R}_n := (\mathbf{r}_0, \ldots, \mathbf{r}_n)^T$,

$$\mathbf{R}(t) = \sum_{i=0}^{n} B_i^n(t)\mathbf{r}_i = \mathbf{B}_n\mathbf{R}_n, \quad t \in [0, 1], \tag{2}$$

such that the L_2 distance

$$E = \sum_{k=1}^{K} \int_{t_k}^{t_{k+1}} \left\| \mathbf{R}(t) - \mathbf{P}^k\left(\frac{t-t_k}{\Delta t_k}\right) \right\|^2 \mathrm{d}t = \sum_{k=1}^{K} \Delta t_k \int_0^1 \|\mathbf{B}_n S_k \mathbf{R}_n - \mathbf{B}_{n_k}\mathbf{P}_{n_k}^k\|^2 \mathrm{d}t \tag{3}$$

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is minimized. Here, $0 = t_1 < t_2 < \cdots < t_{K+1} = 1$ is a partition of the interval [0, 1], $\Delta t_k = t_{k+1} - t_k$, and $S_k\mathbf{R}_n$ represents the control points of the sub-curve $\mathbf{R}(t), t \in [t_k, t_{k+1}]$ after it is reparametrized to the standard interval [0, 1]. See [1] for the expression of the $(n+1) \times (n+1)$ matrix S_k.

In geometric modeling, a complex shape is generally constructed by using multiple segments, with the advantage of producing more desirable shapes through local adjustment. It is often required to merge multiple adjacent segments into a single one, while preserving as faithful as possible to the original shape. For this goal, the merging problem is studied as a minimization problem using some suitable distances such as (3).

To meet certain smoothness requirement in various applications, it is important to impose continuity conditions on the two endpoints of the merged curve $\mathbf{R}(t)$, which correspond to the initial endpoint of the first segment $\mathbf{P}^1(t)$ and the terminal endpoint of the last segment $\mathbf{P}^K(t)$. As multiple segments are merged at a time, multiwise merging does not need to consider continuity conditions at the interior endpoints. It is well known [3,4] that in contrast to the classical parametric continuity (C^k), geometric continuity (G^k) shares additional parameters that are available for further shape manipulation by direct assignment or optimization techniques. Noting the fact that geometric continuity is a relaxation of parametrization (but not a relaxation of smoothness), it is more suitable to use geometric continuity for the merging problem since the parametrization of unknown merged curve should not be simply taken as that of the first (or last) segment. For instance, G^2 continuity (mostly used in applications) is able to retain the positions, tangent directions and curvatures.

1.1. Related work

Most of the existing merging approaches can be categorized into *matrix-based approaches*. The merging problem was firstly introduced in 1987 by Hoschek [5], where many segments of curves are approximately converted into a spline curve through degree reduction and splitting. Although degree reduction was widely investigated by many researchers, merging has not received adequate attention until Hu et al. [6] proposed a pioneering work of pairwise merging where a pair of Bézier curves is merged into one segment instead of applying degree reduction twice. Then, merging has become a special research topic. In terms of matrix representations and operations, many approaches have arisen, including pairwise merging of B-spline curves [7] and Bézier curves [8,9], and multiwise merging of Bézier curves [10,1] and B-spline surfaces [11]. In particular, $C^{r,s}$ multiwise merging in [1] can preserve C^r continuity and C^s continuity at the two endpoints respectively.

A novel *dual approach* was proposed by Woźny et al. [2] for $C^{r,s}$ multiwise merging, which makes use of the advantages of constrained dual Bernstein basis [12,13]. Remarkably, this approach has lower complexity due to the fast evaluation schemes of certain connections between Bernstein and dual Bernstein polynomials.

1.2. Contributions

In this paper, we revisit the multiwise merging problem with the constraint of G^2 or G^3 continuity, and propose an iterative algorithm based on the projected Newton method. According to the G^2 constraint, there are four parameters (i.e., α_1, α_2, β_1, β_2 in (5)–(8)), and two more parameters by the G^3 constraint. Similar to the degree reduction work of Lu [14], we also find that for G^2 multiwise merging, α_2 and β_2 are both expressed as quadratic polynomials of α_1 and β_1 when the minimum of the L_2 distance (3) is reached; therefore, the number of independent parameters is reduced to 2, and the L_2 distance is a quartic polynomial in two unknowns.

It is important to note that many geometric modeling applications often require the two parameters α_1 and β_1 to be positive, so that the end tangents of the merged curve coincide with the specified directions, referring to (5) and (7). On the other hand, the values of α_1 and β_1 indicate the magnitudes of the end tangents. For these reasons, we impose a *feasible region* on (α_1, β_1):

$$\mathcal{D} = \left\{(\alpha_1, \beta_1) \in \mathbb{R}^2 : 0 < l_0 \le \alpha_1 \le u_0,\ 0 < l_1 \le \beta_1 \le u_1\right\}. \quad (4)$$

By restricting (α_1, β_1) in the feasible region, the merging problem is changed to a constrained minimization problem which may be efficiently solved by the projected Newton method, as done in [14] for degree reduction.

Our contributions are summarized as follows:

- An iterative algorithm is proposed for G^2 multiwise merging with the feasible region.
- An intuitive interaction tool is provided for shape-preserving merging, in terms of the feasible region.

Compared to the previous $C^{r,s}$ multiwise merging [1,2], our method produces better results due to the used G^2 and G^3 constraints. Our method can be regarded as a generalization of the previous pairwise merging [8] which uses C^1G^2 and C^1G^3 constraints. Very recently, Gospodarczyk and Woźny [15] proposed a new method for $C^{r,s}$ multiwise merging by restricting all the interior control points of the merged curve in a specified region; whereas our method restricts (α_1, β_1) in a specified region.

2. Iterative algorithm for G^2 multiwise merging

Given K segments of adjacent Bézier curves $\mathbf{P}^k(t)$ with control points $\mathbf{P}^k_{n_k}$, $k = 1, \ldots, K$, G^2 multiwise merging amounts to identifying the control points $\mathbf{R}_n$ of the merged curve $\mathbf{R}(t)$, such that the L_2 distance is minimized and the G^2 constraint at the endpoints is satisfied.