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A mass conservative Generalized Multiscale Finite Element Method applied to two-phase flow in heterogeneous porous media

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ABSTRACT

In this paper, we propose a method for the construction of locally conservative flux fields through a variation of the Generalized Multiscale Finite Element Method (GMsFEM). The flux values are obtained through the use of a Ritz formulation in which we augment the resulting linear system of the continuous Galerkin (CG) formulation in the higher-order GMsFEM approximation space. In particular, we impose the finite volume-based restrictions through incorporating a scalar Lagrange multiplier for each mass conservation constraint. This approach can be equivalently viewed as a constraint minimization problem where we minimize the energy functional of the equation restricted to the subspace of functions that satisfy the desired conservation properties. To test the performance of the method we consider equations with heterogeneous permeability coefficients that have high-variation and discontinuities, and couple the resulting fluxes to a two-phase flow model. The increase in accuracy associated with the computation of the GMsFEM pressure solutions is inherited by the flux fields and saturation solutions, and is closely correlated to the size of the reduced-order systems. In particular, the addition of more basis functions to the enriched multiscale space produces solutions that more accurately capture the behavior of the fine scale model. A variety of numerical examples are offered to validate the performance of the method.

1. Introduction and problem statement

In this paper, we primarily consider the equation given by

$$\begin{aligned} -\mathrm{div}(\Lambda k(x)\nabla p) &= q && \text{in } \Omega \\ p &= p_D && \text{on } \Gamma_D \\ -\Lambda k\nabla p\cdot\mathbf{n} &= g_N && \text{on } \Gamma_N \end{aligned} \tag{1}$$

where $k(x)$ is a heterogeneous field with high contrast. In particular, we assume that there is a positive constant $k_{\min}$ such that $k(x) \geq k_{\min} > 0$, while $k(x)$ can have very large values (i.e., $k_{\max}/k_{\min}$ is large). Additionally, Λ is a known mobility coefficient, q denotes any external forcing, and p is an unknown pressure field satisfying Dirichlet and Neumann boundary conditions given by p_D and g_N, respectively. Here Ω a convex polygonal and two dimensional domain with boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$.

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Let us consider a function in $H^1(D)$ whose trace on Γ_D coincides with the given value p_D; we denote this function also by p_D. The variational formulation of problem (1) is to find $p \in H^1(\Omega)$ with $(p - p_D) \in H^1_D = \{w \in H^1(\Omega) : w|_{\Gamma_D} = 0\}$ and such that

$$a(p, v) = F(v) - \langle g_N, v\rangle_{\Gamma_N} \quad \text{for all } v \in H^1_D, \tag{2}$$

where, for $p, v \in H^1(\Omega)$, the bilinear form a is defined by

$$a(p, v) = \int_\Omega \Lambda k(x)\nabla p(x)\nabla v(x)dx, \tag{3}$$

the functional F is defined by

$$F(v) = \int_\Omega q(x)v(x)dx \tag{4}$$

and the linear functional related to the boundary condition is given by

$$\langle g_N, v\rangle_{\Gamma_N} = \int_{\Gamma_N} g_N(x)v(x)dl. \tag{5}$$

A main goal of our work is to obtain conservative discretizations of the equations above. More specifically, construction of approximation that satisfy some given conservation of mass restriction on subdomains of interest. We note that a popular conservative discretization is the Finite Volume (FV) method. The classical FV discretization provides and approximation of the solution in the space of piecewise linear functions with respect to a triangulation while satisfying conservation of mass on elements of a dual triangulation. When the approximation of the piecewise linear space is not enough for the problem at hand, advance approximation spaces need to be used. However, in some cases this requires a sacrifice of the conservation properties of the FV method. In this work we present an extension of the FV method for general approximation spaces that enrich classical approximation spaces (such as the space of piecewise linear functions). In particular, this conservative discretization can be used in conjunction with recently introduced GMsFEM spaces.

We note that FV methods that use higher degree piecewise polynomials have been introduced in the literature. The fact that the dimension of the approximation spaces is larger than the number of restrictions led the researchers of [1,2] to introduce additional control volumes to match the number of restrictions to the number of unknowns. An alternative approach is to consider a Petrov–Galerkin formulation with additional test functions rather that only piecewise constant functions on the dual grid. They where able to obtain stability of the method as well as error estimates. It is important to observe that the additional control volumes require additional computational work to be constructed and in some cases are not easy to construct (see also [3,4]). Additionally, it is well know that piecewise smooth approximations spaces do not perform well for multiscale high-contrast problems [5–12]. Another technique that one can use in order obtain a conservative method with richer approximations spaces is the following (see for instance [13] where the authors use a similar approach). In the discrete linear system obtained by a finite element discretization, it is possible to substitute appropriate number of equations by finite volume equations involving only the standard dual grid. This approach has the advantage that no additional control volume needs to be constructed. It may have the flexibility of both FV and FE procedures given as mass conservative fluxes and residual minimization properties. Some previous numerical experiments suggest a drawback of this approach—the resulting discrete problems may be ill-conditioned for large dimension coarse spaces, especially for higher order approximation spaces and multiscale problems (see [13]).

In this paper we propose the alternative of using a Ritz formulation and construct a solution procedure that combines a continuous Galerkin-type formulation that concurrently satisfies mass conservation restrictions. To this end, we augment the resulting linear system of the Galerkin formulation in the higher order approximation space to impose the finite volume restrictions. In particular, we do that by using a scalar Lagrange multiplier for each restriction. This approach can be equivalently viewed as a constraint minimization problem where we minimize the energy functional of the equation restricted to the subspace of functions that satisfy the conservation of mass restrictions. Then, in the Ritz sense, the obtained solution is the best among all functions that satisfy the mass conservation restriction.

As a main application of the techniques presented here, we consider the case where the coefficient k has high-variation and discontinuities (not necessarily aligned with the coarse grid). For this problem it is known that higher order approximation is needed. Indeed, in some cases robust approximation properties, independent of the contrast, are required. See for instance [8–10] where it is demonstrated that classical multiscale methods [14] do not render robust approximation properties in terms of the contrast. It is shown that one basis functions per coarse node (with the usual support) is not enough to construct adequate coarse spaces [9,15]. A similar issue can be expected for the multiscale finite volume method developed in [16–18] and related works, when applied to high-contrast multiscale problems since the approximation spaces have similar approximation properties. In the case of Galerkin formulations, robust approximation properties are obtained by using the Generalized Multiscale Finite Element Method (GMsFEM) framework. The main goal of GMsFEMs is to construct coarse spaces for Multiscale Finite Element Methods (MsFEMs) that result in accurate coarse-scale solutions. This methodology was first developed in [5–7] based on some previous works [8–12]. A main ingredient in the construction is the use of an approximation of local eigenvectors (of carefully selected local eigenvalues problem) to construct the coarse