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Quintic polynomial approximation of log-aesthetic curves by curvature deviation☆

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ABSTRACT

Log-aesthetic curves (LACs), possessing monotone curvature and including many classical curves, have been widely used to describe fair shapes in geometric modeling. However, they are generally represented in non-polynomial form and are thus not compatible with current CAD systems. In this paper we present quintic polynomial approximation of LAC segments. For a given LAC segment, a quintic G^2 interpolating Bézier curve is obtained by minimizing a curvature-based error metric, with the advantage of being more likely to preserve the monotone curvature property. Numerical experiments demonstrate that our method can usually generate better results than the previous methods in terms of the deviation in positions and curvatures.

1. Introduction

Log-aesthetic curves (LACs) have been widely used to describe visually fair shapes in computer aided design, computer graphics and computer arts [1–4]. This family of curves is defined with linear logarithmic curvature graphs and thus possesses monotone curvature [5,6]:

$$\log\left(\rho\frac{ds}{d\rho}\right) = \alpha\log\rho + c \tag{1}$$

where $c = -\log\lambda$ is a constant, s and ρ are the arc length and the radius of curvature. Log-aesthetic curves include as special cases the Clothoids, Nielsen spirals, logarithmic spirals, circle involutes and circular arcs when $\alpha = -1, 0, 1, 2$ and ∞, respectively.

A LAC segment can be expressed by parametric equations in terms of definite integrals [5,6] and incomplete gamma functions [7]. While evaluating these two kinds of expressions, the problems of computational inefficiency and rounding errors may occur because numerical integration or infinite series is required. Furthermore, log-aesthetic curves are not compatible with nonuniform rational B-splines (NURBS) which are an industry standard in CAD/CAM [8]. Therefore, it is often required to approximate a LAC segment by a (rational) polynomial curve. Many researchers have focused on the approximations of special forms of LACs, such as circular arcs [9–11], logarithmic spirals [12] and Clothoids [13–15]. In particular, the recent quintic interpolation method introduced by Sánchez-Reyes and Chacón [15] shares C^2 quasi arc-length parameterization. Yoshida and Saito [16] presented rational cubic G^2 approximation by minimizing the deviation

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in positions, where a rational cubic curve is constructed by choosing the optimal values for two variables associated with the G^1 constraint.

This paper deals with the problem of quintic polynomial approximation of LAC segments. For a given LAC segment, our aim is to find an approximating quintic G^2 Bézier curve, while preserving the monotone curvature property as much as possible. We limit the curve representation to quintics instead of cubics or quartics, since a quintic curve with four unconstrained variables can match G^2 Hermite data consisting of positions, tangents and curvatures. Another reason is that the rational form may suffer from a few annoying problems in engineering design, as pointed out by Piegl et al. [11]. On the other hand, we adopt a curvature-based error metric to preserve the monotone curvature property of a LAC segment by penalizing curvature deviation. As is well known, the curvature field of a planar curve parameterized by arc length fully prescribes it up to a rigid motion transformation. Thus, when the curvature profile of a quintic G^2 interpolating curve is as similar as possible to that of the LAC segment, the quintic curve is also a satisfactory approximation in the sense of position deviation.

Our method is mostly closely related to the previous methods [15,16]. Compared to the quintic C^2 quasi arc-length method [15], we represent the quintic curve with four variables (whose values are determined by optimization). Compared to the rational cubic G^2 method [16], we adopt the quintic polynomial representation with two more degrees of freedom and pay special attention to the deviation in curvatures.

2. Preliminaries: log-aesthetic curves

A log-aesthetic curve is characterized by the fundamental equation (1), and its *standard form* in terms of the tangent angle θ is derived in [6,7]:

$$\mathbf{P}(\theta) = \left(\int_0^\theta \rho(\psi)\cos\psi\, d\psi,\ \int_0^\theta \rho(\psi)\sin\psi\, d\psi\right) \tag{2}$$

where the relation between ρ (or curvature κ) and θ is

$$\rho(\theta) = \frac{1}{\kappa(\theta)} = \begin{cases} e^{\lambda\theta}, & \text{if } \alpha = 1, \\ ((\alpha-1)\lambda\theta + 1)^{\frac{1}{\alpha-1}}, & \text{otherwise.} \end{cases} \tag{3}$$

This standard form is achieved such that a reference point on the curve is placed at the origin, with the tangent parallel to the x-axis and $\rho = 1$ when $\theta = 0$. Noticeably, κ decreases as θ increases.

The arc length of $\mathbf{P}(\theta)$ is expressed as

$$s(\theta) = \int_0^\theta \rho(\psi)d\psi = \begin{cases} -\frac{1}{\lambda}\log(1-\lambda\theta), & \text{if } \alpha = 0, \\ \frac{1}{\lambda}(e^{\lambda\theta}-1), & \text{if } \alpha = 1, \\ \frac{1}{\lambda\alpha}\left(((\alpha-1)\lambda\theta+1)^{\frac{\alpha}{\alpha-1}} - 1\right), & \text{otherwise.} \end{cases} \tag{4}$$

Based on the inverse of (4), it is available to parameterize the log-aesthetic curve in arc length, i.e., $\mathbf{P}(s^{-1}(s))$, but which is not analytical. For more details on the definitions and properties of log-aesthetic curves, see [5–7].

3. Approximating a LAC segment by a quintic Bézier curve

We consider the problem of approximating a LAC segment, $\mathbf{P}(\theta), \theta \in [\theta_0, \theta_1]$, by a quintic curve. Assume that the quintic curve is expressed in Bernstein–Bézier form

$$\mathbf{Q}(t) = \sum_{i=0}^{5} B_i^5(t)\mathbf{q}_i = \sum_{i=0}^{5} \frac{5!}{i!(5-i)!} t^i(1-t)^{5-i}\mathbf{q}_i, \quad t \in [0,1], \tag{5}$$

where $\mathbf{q}_i$ are the control points.

To interpolate positions, tangents and curvatures of a LAC segment, the quintic curve must satisfy the G^2 constraint at the two endpoints. That is, when $t = 0, 1$, we have (see [17])

$$\begin{aligned} &\mathbf{Q}(t) = \mathbf{P}(\phi(t)), \\ &\mathbf{Q}'(t) = \phi'(t)\mathbf{P}'(\phi(t)), \\ &\mathbf{Q}''(t) = \left(\phi'(t)\right)^2 \mathbf{P}''(\phi(t)) + \phi''(t)\mathbf{P}'(\phi(t)), \end{aligned}$$

where $\theta = \phi(t) : [0, 1] \to [\theta_0, \theta_1]$ is some suitable reparameterization.