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Asymptotic mean-square stability of explicit Runge–Kutta Maruyama methods for stochastic delay differential equations

Qian Guo [a,*], Mingming Qiu [a], Taketomo Mitsui [b]

[a] Department of Mathematics, Shanghai Normal University, 100 Guilin Road, Shanghai 200234, China

[b] Nagoya University, Nagoya, Japan

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ABSTRACT

As relatively little is known about Runge–Kutta type method applied to stochastic delay differential equations, we present an explicit Runge–Kutta Maruyama (RKM) method for solving them. The mean-square stability properties of the numerical solutions generated by the RKM method are investigated, and a sufficient condition for stability is obtained and applied to the S-ROCK type methods for stochastic delay differential equations. Numerical examples are provided to confirm theoretical results.

1. Introduction

Stochastic delay differential equations (SDDEs) have attracted an increasing interest in many different disciplines, especially in biology, see, *e.g.*, [1–3]. Usually analytic solutions for SDDEs are hardly available. In this case the importance of numerical methods is obvious. Several types of numerical methods have been introduced to solve SDDEs, e.g. Euler–Maruyama methods [4–6], multi-step Maruyama methods [7], Milstein methods [8], and Wong–Zakai approximation [9].

Numerical stability plays an important role in numerical analysis. Stability theory for numerical simulations of SDDEs typically deals with mean-square behavior. For example, Baker and Buckwar [10] developed a pth mean stability analysis of the Euler–Maruyama type methods for a linear SDDE. Liu et al. [11] studied the mean-square stability of the stochastic theta method for a linear scalar SDDE. Huang et al. [12] studied the delay dependent stability of the stochastic theta method. Wang and Gan [13] investigated the mean-square exponential stability of a split-step Euler method. Wang and Zhang [14] obtained some stability conditions for the Milstein method. We note that nonlinear stability of numerical methods, including Euler-type and the theta methods, has also received attention (see, e.g., [15–17]). In recent years there has been a growing interest in almost sure stability of numerical methods applied to SDDEs. Wu and his colleagues found the almost sure stability conditions of the solutions of Euler-type schemes to SDDEs by using the discrete semimartingale convergence theorem (see, for example, [18,19]).

* Corresponding author.
E-mail addresses: qguo@shnu.edu.cn (Q. Guo), tom.mitsui@nagoya-u.jp (T. Mitsui).

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As the major family of numerical methods, the Runge–Kutta methods have been widely used in temporal discretization for the approximation of solutions of differential equations. See the monograph [20–22] for a review and further references. To the best of our knowledge, the Runge–Kutta method for SDDEs is rarely mentioned in the literature in spite of considerable interests to that for SODEs in, *e.g.*, [23–30]. In deterministic case, one can improve the convergence rate by increasing the stage number of a Runge–Kutta type method, however, in the stochastic case, the convergence rate also depends on the discretization of the stochastic integrals, see, for example, [29,30]. Note that Hu et al. [8] proposed an approach for simulating the multiple stochastic integrals with time-delay, but it is unfeasible to achieve first order because one needs a great many terms in the Karhunen-Loéve expansion for a small time step-size (see, [8, Appendix B]). This paper is devoted to present a family of stochastic Runge–Kutta (SRK) methods for SDDEs without deployment of multiple stochastic integrals, which is called the Runge–Kutta Maruyama (RKM) method.

On the other hand, comparing with traditional explicit methods, implicit methods have better stability properties, but this comes with implementation costs dominated by the iteration scheme required to solve the implicit stage equations at each time step. Meanwhile, stabilized explicit Runge–Kutta schemes have proved successful for solving SODEs, which are called S-ROCK (stochastic orthogonal Runge–Kutta–Chebyshev) method. We refer the reader to the papers of Abdulle and his colleagues [23–25], Komori and Burrage [27,28] for more details on the S-ROCK method. We note that the authors of [31,32] performed a mean-square stability analysis for several numerical methods applied to test systems of SODEs. Another approach to numerical stability analysis for the SODE system is recently developed by Buckwar and Sickenberger [33]. It is interesting to further develop an approach to analyze the stability of numerical scheme for solving SDDEs. Then, our main aim is to present the explicit Runge–Kutta Maruyama method for solving SDDE system and to investigate its stability criteria. Here we will also focus on stability analysis of the S-ROCK type methods applied to the SDDE system.

Consider the Itô SDDEs on the time interval $[0, T]$ with delay $\tau > 0$ given by

$$\begin{cases} dX = F(X(t), \quad X(t-\tau))\,dt + G(X(t), \quad X(t-\tau))\,dw(t), \\ X(t) = \psi(t) \quad \text{for } t \le 0, \end{cases} \tag{1}$$

where X, ψ, F and G are ℓ-dimensional column vector functions. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge 0}$ satisfying the usual conditions, that is, it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. Let $w(t)$ be the scalar Wiener process defined on this probability space.

Unless otherwise specified, we use the following notations. The symbol $\lfloor\cdot\rfloor$ denotes the floor function, that is, for $x \in \mathbb{R}$, $\lfloor x\rfloor$ gives the largest previous integer. The unit matrix of ℓ-dimension is denoted by I_ℓ. Let $|\cdot|$ be the L^2 norm in $\mathbb{R}^\ell$. Throughout this paper, c is a generic constant that depends on F, G, the initial data ψ, the interval of integration $[0, T]$, but is independent of the discretization parameter and choice of time point $t \in [0, T]$.

If A is a vector or matrix, its transpose is denoted by $A^\top$.

To existence and uniqueness of the solution of SDDEs (1), we need the following assumptions (see, [34, Section 5.3]).

A1. The functions F and G satisfy the uniform Lipschitz condition

$$|F(x_1, y_1) - F(x_2, y_2)|^2 \bigvee |G(x_1, y_1) - G(x_2, y_2)|^2 \le c(|x_1 - x_2|^2 + |y_1 - y_2|^2) \tag{2}$$

for $t \in [0, T]$ and any $x_1, x_2, y_1, y_2 \in \mathbb{R}^\ell$, where $\bigvee$ is the maximal operator.

A2. The functions F and G satisfy the linear growth condition

$$|F(x, y)|^2 \bigvee |G(x, y)|^2 \le c(1 + |x|^2 + |y|^2) \tag{3}$$

for $t \in [0, T]$ and any $x, y \in \mathbb{R}^\ell$.

A3. The initial function $\psi \in C([-\tau, 0], \mathbb{R}^\ell)$ is Lipschitz continuous from $[-\tau, 0]$ to $\mathbb{R}^\ell$.

The rest of this paper is organized as follows. In Section 2, we present an explicit Runge–Kutta Maruyama method and show its convergence order. In Section 3, the numerical stability analysis of a linear system is performed and a stability criterion is derived. Moreover, the S-ROCK type methods are proposed for SDDEs and the stability criterion is applied to these methods in Section 4. In Section 5, some numerical examples are given to confirm the theory. In the last section we draw our conclusions.

2. The Runge–Kutta Maruyama method for SDDEs

In the following, we employ an equidistant step-points $\mathcal{I}_{\Delta t} = \{t_0, t_1, \ldots, t_N\}$ where the time step-size is equal to $\Delta t = \tau/m$ for a given positive integer m, and the nth step-point is denoted by $t_n = n\,\Delta t$ for $0 \le n \le N$. The numerical approximation of $X(t)$ at t_n is denoted by Y_n.

We consider the following family of ν-stage RKM methods for SDDEs (1), defined by

$$Y_{n+1} = Y_n + \Delta t \sum_{i=1}^{\nu} b_i F\left(K_{n,i}, \tilde{K}_{n,i}\right) + \sum_{i=1}^{\nu} d_i G\left(K_{n,i}, \tilde{K}_{n,i}\right) \Delta W_n \tag{4}$$