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Generalizations of Darbo's fixed point theorem via simulation functions with application to functional integral equations

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ABSTRACT

In this paper, firstly, we introduce Z_μ-contraction and $\Im_\mu$-contraction via stimulation functions. Secondly, we prove some new fixed point theorems for Z_μ-contractive mappings and $\Im_\mu$-contractive mappings. Our results generalize and extend some existing results. Moreover, some examples and an application to functional integral equations are given to support the obtained results.

1. Introduction and preliminaries

Schauder fixed point theorem is one of the most fruitful and effective tools in nonlinear analysis. In 1955, Darbo [1], using the concept of a measure of non-compactness, proved the fixed point property for α-set contraction on a closed, bounded and convex subset of Banach spaces. Darbo's fixed point theorem is a significant extension of the Schauder fixed point theorem, and it also plays a key role in nonlinear analysis especially in proving the existence of solutions for a lot of classes of nonlinear equations. Since then, some generalizations of Darbo's fixed point theorem have been proved. For example, we refer the reader to [2–7] and the references therein. Recently, Aghajani et al. [8] and Cai et al. [9] proved some new generalizations of Darbo's fixed point theorem.

Throughout this paper, by $\mathbf{N}$, $\mathbf{R}_+$ and $\mathbf{R}$, respectively, denote the set of all positive integers, non-negative real numbers and real numbers.

Now, let us recall some basic concepts, notations and known results which will be used in the sequel. In this paper, we let E be a Banach space with the norm $\|\cdot\|$ and θ be the zero element in E. The closed ball centered at x with radius r is denoted by $B(x, r)$, by simply B_r if $x = 0$. If X is a nonempty subset of E, then we denote by $\overline{X}$ and $\mathrm{Conv}(X)$ the closure and closed convex hull of X, respectively. Moreover, let $\mathfrak{M}_E$ be the family of all nonempty bounded subsets of E and by $\mathfrak{N}_E$ the subfamily consisting of all relatively compact subsets of E.

In [10], Banaś et al. gave the concepts of a measure of non-compactness.

Definition 1.1. A mapping $\mu : \mathfrak{M}_E \to \mathbf{R}_+$ is said to be a measure of non-compactness in E if it satisfies the following conditions:

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(1) The family $\ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker\mu \subseteq \mathfrak{N}_E$;
(2) $X \subseteq Y \Rightarrow \mu(X) \leq \mu(Y)$;
(3) $\mu(\overline{X}) = \mu(X)$;
(4) $\mu(ConvX) = \mu(X)$;
(5) $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ for all $\lambda \in [0, 1]$;
(6) If $\{X_n\}$ is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subseteq X_n$ for $n = 1, 2, \ldots$ and $\lim_{n\to\infty}\mu(X_n) = 0$, then the intersection set $X_\infty = \cap_{n=1}^{\infty} X_n$ is nonempty.

The family $\ker\mu$ described in (1) is said to be the kernel of the measure of non-compactness μ. Observe that the intersection set X_∞ from (4) is a member of the family $\ker\mu$. In fact, since $\mu(X_\infty) \leq \mu(X_n)$ for any n, we infer that $\mu(X_\infty) = 0$. This yields that $X_\infty \in \ker\mu$.

Theorem 1.1 (*Schauder Fixed Point Principle*)**.** *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E. Then each continuous and compact map $T : \Omega \to \Omega$ has at least one fixed point in the set Ω.*

Obviously the above formulated theorem constitutes the well known Schauder fixed point principle. Its generalization, called the Darbo's fixed point theorem, is formulated below.

Theorem 1.2 (*Darbo's Fixed Point Theorem [1]*)**.** *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and let $T : \Omega \to \Omega$ be a continuous mapping. Assume that there exists a constant $k \in [0, 1)$ such that*

$$\mu(TX) \leq k\mu(X)$$

for any nonempty subset X of Ω, where μ is a measure of non-compactness defined in E. Then T has a fixed point in the set Ω.

In order to prove our fixed point theorems, we need some of the following related concepts. First of all, we recall the definition of the class of function as follows.

Definition 1.2 (*Khan et al. [11]*)**.** An altering distance function is a continuous, nondecreasing mapping $\phi : [0, \infty) \to [0, \infty)$ such that $\phi^{-1}(\{0\}) = \{0\}$.

The notions of simulation function was introduced by Khojasteh et al. in [12] as follows.

Definition 1.3. A simulation function is a mapping $\xi : [0, \infty) \times [0, \infty) \to \mathbf{R}$ satisfying the following conditions:
(ξ_1) $\xi(0, 0) = 0$;
(ξ_2) $\xi(t, s) < s - t$ for all $t, s > 0$;
(ξ_3) if $\{t_n\}$, $\{s_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, then

$$\limsup_{n\to\infty} \xi(t_n, s_n) < 0.$$

However, in [12], the authors slightly modified the definition of simulation function which introduced by Khojasteh et al. [13] and enlarged the family of all simulation functions.

Definition 1.4. A simulation function is a mapping $\xi : [0, \infty) \times [0, \infty) \to \mathbf{R}$ satisfying the following conditions:
(ξ_1) $\xi(0, 0) = 0$;
(ξ_2) $\xi(t, s) < s - t$ for all $t, s > 0$;
(ξ_3) if $\{t_n\}$, $\{s_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ and $t_n < s_n$, then

$$\limsup_{n\to\infty} \xi(t_n, s_n) < 0.$$

Let Z be the family of all simulation functions $\xi : [0, \infty) \times [0, \infty) \to \mathbf{R}$ in Definition 1.4.

Thereupon, the authors [13] give the following example to illustrate that every simulation function in the original Khojasteh et al.'s sense (Definition 1.3) is also a simulation function in Roldán-L'pez-de-Hierro et al.'s sense (Definition 1.4), but the converse is not true.

Example 1.1. Let $k \in \mathbf{R}$ be such that $k < 1$ and let $\xi : [0, \infty) \times [0, \infty) \to \mathbf{R}$ be the function defined by

$$\xi(t, s) = \begin{cases} 2(s-t) & \text{if } s < t \\ ks - t & \text{otherwise.} \end{cases}$$

Clearly, ξ verifies (ξ_1), and (ξ_2) follows from

$$t, s > 0, \quad \begin{cases} 0 < s < t \Rightarrow \xi(t, s) = 2(s-t) < s - t, \\ 0 < s \leq t \Rightarrow \xi(t, s) = ks - t < s - t. \end{cases}$$