Journal of Computational and Applied Mathematics 296 (2016) 638–648

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Marshall–Olkin type copulas generated by a global shock

Fabrizio Durante [a,*], Stéphane Girard [b], Gildas Mazo [c]

[a] Faculty of Economics and Management, Free University of Bozen-Bolzano, Bolzano, Italy
[b] Inria Grenoble Rhône-Alpes and Laboratoire Jean Kuntzmann, France
[c] ISBA - Institut de Statistique, Biostatistique et Sciences Actuarielles, Université Catholique de Louvain, Louvain-la-Neuve, Belgium

ARTICLE INFO

Article history:
Received 30 March 2015
Received in revised form 28 September 2015

MSC:
62H05
62G32

Keywords:
Copula
Marshall–Olkin distributions
Shock models

ABSTRACT

A way to transform a given copula by means of a univariate function is presented. The resulting copula can be interpreted as the result of a global shock affecting all the components of a system modeled by the original copula. The properties of this copula transformation from the perspective of semi-group action are presented, together with some investigations about the related tail behavior. Finally, the whole methodology is applied to model risk assessment.

1. Introduction

Starting with the seminal paper by Marshall and Olkin [1], Marshall–Olkin distributions (and copulas) have been extensively exploited for modeling multivariate random vectors. As is known, these distributions arise from an intuitive interpretation in terms of shock models. In fact, a random vector is said to follow a Marshall–Olkin distribution if its components are interpreted as future failure times which are defined as the minimum of independent, exponential arrival times of exogenous shocks.

Starting with these ideas, different extensions of Marshall–Olkin distributions have been provided in the literature by supposing, for instance, that the shocks follow specific distributions or fail to be independent. See, for instance, [2,3] and references therein and recent contributions by Li and Pellerey [4]; Kundu et al. [5]; Lin and Li [6]; Ozkut and Bayramoglu [7]. Durante et al. [8] calls *Marshall–Olkin machinery* the common stochastic mechanism that drives many of these extensions.

Here we are interested in a Marshall–Olkin-type copula generated by a simple mechanism. Given a set of (continuous) random variables with copula C, we assume that their common behavior is modified by a shock that affects all the variables at the same time. This results in a modification of the copula C by means of a function f, which depends on the shock distribution. Despite its simplicity, this modification has several advantages since, for instance, it allows to generate models with various tail dependences and singularities, as will be clarified in the sequel.

Specifically, in Section 2 we present the basic properties of this model and their connections with several results already presented in the literature. Section 3 focuses on the interpretation of this transformation as action of a semigroup of real-valued functions on the class of copulas. The tail behavior induced by the transformation is considered in Section 4. Section 5 illustrates a possible application in the framework of model risk assessment.

* Corresponding author.
E-mail addresses: fabrizio.durante@unibz.it (F. Durante), stephane.girard@inria.fr (S. Girard), gildas.mazo@uclouvain.be (G. Mazo).

http://dx.doi.org/10.1016/j.cam.2015.10.022
0377-0427/

2. The model and its first properties

In the following, we use standard definitions and properties of copulas, as they can be found, for instance, in [9–11].

Let $\mathscr{F}$ be the class of increasing and continuous functions $f : [0, 1] \to [0, 1]$ such that $f(1) = 1$ and id/f is increasing, where id denotes the identity function on [0, 1]. The elements of $\mathscr{F}$ are *anti-star-shaped* functions (see, e.g., [12,13]), i.e. they are characterized by the property $f(\alpha t) \geq \alpha f(t)$ for every $\alpha \in [0, 1]$. Moreover, if $f \in \mathscr{F}$, then for all $t \in [0, 1]$, $f(t)/t \geq f(1)/1 = 1$, from which it follows that $f(t) \geq t$ for all $t \in [0, 1]$.

Let $\mathscr{C}_d$ be the set of all d-variate copulas.

Definition 2.1. For all $f \in \mathscr{F}$ and $C \in \mathscr{C}_d$, the function $T(f, C) : [0, 1]^d \to [0, 1]$ given, for every $(u_1, \ldots, u_d) \in (0, 1]^d$, by

$$T(f, C)(u_1, \ldots, u_d) = C(f(u_1), \ldots, f(u_d)) \frac{\min(u_1, \ldots, u_d)}{f(\min(u_1, \ldots, u_d))}, \tag{2.1}$$

while $T(f, C)(u_1, \ldots, u_d) = 0$ if $u_i = 0$ for at least one index $i \in \{1, \ldots, d\}$, is called *shock transformation* of C via f.

Since $T(f, C)$ can be rewritten as

$$T(f, C)(u_1, \ldots, u_d) = C(f(u_1), \ldots, f(u_d)) M_d(g(u_1), \ldots, g(u_d))$$

for all $(u_1, \ldots, u_d) \in [0, 1]^d$ with $f \cdot g = \mathrm{id}$, and $M_d(u_1, \ldots, u_d) = \min(u_1, \ldots, u_d)$ is the Hoeffding–Fréchet upper bound, it can be interpreted as a particular case of the construction method introduced in [14, Theorem 2.1]. Thus, the following result easily follows.

Proposition 2.1. *For all $f \in \mathscr{F}$ and $C \in \mathscr{C}_d$, $T(f, C)$ is a copula.*

Copula models of type (2.1) extend the bivariate dual extended Marshall–Olkin model by Pinto and Kolev [15] to the multivariate framework. Moreover, they can be also interpreted as a particular case of the construction principle considered by Durante et al. [16]. In particular, from the latter reference, the following stochastic interpretation can be derived.

Proposition 2.2. *Let Y be a random variable whose distribution function is g and let $(\tilde{X}_1, \ldots, \tilde{X}_d)$ be any random vector with copula C independent of Y. Denote by $\tilde{F}_1, \ldots, \tilde{F}_d$ the univariate marginal distributions corresponding to $\tilde{X}_1, \ldots, \tilde{X}_d$ and define $f = \mathrm{id}/g$. If $X_i = f^{-1}(\tilde{F}_i(\tilde{X}_i))$ then $T(f, C)$ of Eq. (2.1) is the distribution function of the random vector $(Z_1, \ldots, Z_d)$ such that $Z_i = \max\{X_i, Y\}$ for every $i \in \{1, 2, \ldots, d\}$.*

Proof. Since, under independence assumptions, we can write

$$\mathbb{P}(Z_1 \leq u_1, \ldots, Z_d \leq u_d) = \mathbb{P}(X_1 \leq u_1, \ldots, X_d \leq u_d) \cdot \mathbb{P}(Y \leq \min(u_1, \ldots, u_d)),$$

the result follows from the given assumptions. □

The latter result provides a useful and easy-to-implement algorithm for generating copulas of type (2.1) once an algorithm for generating the starting copula C is available.

It can be easily proved that, for all $f \in \mathscr{F}$ and $C \in \mathscr{C}_d$:

$$\begin{aligned} &T(\mathrm{id}, C) = C, \qquad T(1, C) = M_d, \\ &T(f, M_d) = M_d, \qquad T(f, \Pi_d) = C_f, \end{aligned} \tag{2.2}$$

where $\Pi_d(u_1, \ldots, u_d) = u_1 \ldots u_d$ is the independence copula, while C_f is the copula introduced by Durante et al. [17] and defined by:

$$C_f(u_1, \ldots, u_d) = u_{(1)} \prod_{i=2}^{d} f(u_{(i)}) \tag{2.3}$$

where $u_{(1)}, \ldots, u_{(d)}$ denotes the order statistics of $(u_1, \ldots, u_d) \in [0, 1]^d$. See also [18–22] for the bivariate case.

The stochastic mechanism at the basis of Eq. (2.1) generally produces a copula $T(f, C)$ that has a singular component provided that $f \neq \mathrm{id}$. In fact, in such a case, the first derivatives $T(f, C)$ have jumps. The singular component can be easily computed in the bivariate case.

Proposition 2.3. *Let f be in $\mathscr{F}$ such that both f and $g = \mathrm{id}/f$ are absolutely continuous. Let $C \in \mathscr{C}_2$ be absolutely continuous. Then, the singular component of $T(f, C)$ is given by*

$$S(x, y) = \int_0^{\min(x,y)} g'(u) C(f(u), f(u)) du,$$

for all $(x, y) \in [0, 1]^2$.