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A novel numerical approach and its convergence for numerical solution of nonlinear doubly singular boundary value problems

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ABSTRACT

This paper deals with two new recursive numerical schemes for the solution of a class of doubly singular two-point boundary value problems with different types of boundary conditions. We first transform the original problem into an equivalent integral equation to overcome the singular behaviour at the origin and then establish recursive schemes by employing Homotopy perturbation method for the solution of the resultant equation. We establish the convergence results of the new numerical methods. Four numerical examples are considered to demonstrate the efficiency and accuracy of the proposed methods. The present methods are shown their advantage over some existing methods (Chawla and Katti, 1982; Pandey and Singh, 2007; Pandey and Singh, 2004; Caglar et al., 2009; Kumar and Aziz, 2004; Khuri and Sayfy, 2010) developed for the solution of singular two-point boundary value problems. Moreover, the new methods are used to obtain the approximate solutions of the singular boundary value problems arising in various physical problems.

1. Introduction

We consider the doubly singular two-point boundary value problems of the form

$$(g(x)y')' = q(x)f(x, y), \quad 0 < x < 1, \tag{1}$$

$$y(0) = A, \qquad \mu y(1) + \sigma y'(1) = B, \tag{2}$$

or

$$y'(0) = 0, \qquad \mu y(1) + \sigma y'(1) = B, \tag{3}$$

where $\mu > 0$, $\sigma \geq 0$, and A and B are finite constants. Here $g(0) = 0$, and if $q(x)$ is allowed to be discontinuous at $x = 0$ then the problem (1) is called the doubly singular [1]. The following conditions have been imposed on the functions $g(x)$, $q(x)$ and $f(x, y)$:

C1: $f(x, y)$ is continuous for all $(x, y) \in \{[0, 1] \times \Re\}$,
C2: $f(x, y)$ is continuously differentiable with respect to y, for all $x \in [0, 1]$, and all real y,
C3: $\partial f(x, y)/\partial y \geq 0$,

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C4: $g(x) \geq 0$ in $(0, 1]$ and $g(0) = 0$,
C5: $g(x) \in C[0, 1] \bigcap C^1[0, 1]$,
C6: $1/g(x) \in L_1(0, 1)$,
C7: $q(x) \geq 0$ in $(0, 1]$,
C8: $q(x) \in C^1[0, 1]$,
C9: $q(x) \in L_1(0, 1)$.

Pandey [2–4] has established the existence and uniqueness of the solution to the problems (1) with $g(x) = q(x)$ and boundary conditions $y(0) = A$ (or, $y'(0) = 0$) and $y(1) = B$ using a monotone iterative method. Further, Pandey and Verma [5,6] have established the existence and uniqueness of the solution to the problems (1) with boundary conditions (2) and with boundary conditions (3).

During the last few years, extensive research has been devoted to the study of singular two-point boundary value problems for ordinary differential equation. To some extent, this is due to their importance in applications in various branches of science and technology. They arise in the study of electro hydrodynamics [7], in the theory of thermal explosions [8], generalized axially symmetric potentials in a rectangle [9] and transport process [10]. Eq. (1) with $g(x) = 1$, $q(x) = x^{-1/2}$ and $f(x, y) = y^{3/2}$, and boundary conditions (2) with $\mu = 1$, $\sigma = 0$, is known as Thomas–Fermi equation [11,12]. This equation allows one to calculate the electric potential of the atom. The boundary value problem (1) with $g(x) = q(x) = x^2$ and $f(x, y) = \delta y/(y + \lambda)$, $\delta > 0, \lambda > 0$ arises in the study of steady state oxygen-diffusion in a spherical cell with Michaelis–Menten uptake kinetics [13,14]. Problem (1) with $f(x, y) = \rho e^{-\rho\lambda y}$, $\rho > 0, \lambda > 0$ arises in the study of distribution of heat sources in the human head [15,16]. Furthermore, the problem (1) with $g(x) = q(x) = 1, x$ or x^2, arise in the study of various tumour growth problems with linear function $f(x, y)$, [17–21].

In most cases it is not possible to solve analytically the singular boundary value problems. Various numerical— and analytical approximate techniques such as finite difference method [22–28], spline method [29–31], finite element method [32], differential quadrature method [33], collocation method [34] and Adomian decomposition method (ADM) [35], modified Adomian decomposition method (MADM) [36], Homotopy analysis method (HAM) [37], Variational iteration method (VIM) [38] have been developed for the solution of singular boundary value problems. However, these methods applied to solve the problem (1) with $q(x) = 1$ or $g(x) = q(x)$. The author of [39] has proposed some projection methods to solve linear doubly singular boundary value problems.

In this paper we introduce two new efficient recursive numerical techniques based on HPM for the approximation of a class of nonlinear doubly singular boundary value problems (1)–(3). In particular, we solve three singular boundary value problems that arise in various physical problems: the first problem is an application of nonlinear heat-conduction model in the human head and second problem is an application of oxygen diffusion in a spherical shell, while third problem arises in the study of radial stress on a rotationally symmetric shallow membrane cap.

HPM is a combination of the classical perturbation method and the homotopy concept as used in topology. The method was originally introduced by J-H He [40]. HPM has been effectively applied to solve linear or nonlinear differential equations, fractional differential equations and integral equations [41–52]. The main feature of the method is the condition of homotopy by introducing an embedment parameter which takes the value from 0 to 1. If $p = 0$ the homotopy equation generally reduces to a sufficiently simplified form, which yields a rather simple solution. While $p = 1$ it turns out to be the original problem, and gives the required solution. Unlike the existing recursive schemes using ADM, HAM or modified ADM, this method does not require the computation of undetermined coefficients. Moreover, this method does not require any discretization or linearization of variables as compared to other methods such as finite difference method, finite element method or spline method. The present method requires less computational work as compared to other existing recursive schemes. The approximate solution is obtained in the form of power series with easily calculable components.

We note that the problem (1) has a singularity at the origin. To remove the singularity we first transform the original problem (1) into an integral equation and employ afterwards HPM for solving the resultant equation. We employ the boundary condition at $x = 1$ to eliminate the undetermined coefficient that associated with the method. The convergence analysis of the proposed methods are discussed. To illustrate the effectiveness and accuracy of the methods we apply them on four numerical examples. The numerical results are compared with that obtained using the three-point finite difference methods. Comparison shows that our method provides better solution than the methods given in [22,25,53].

This article is organized as follows. In Section 2, we discuss the basic principles of HPM. In Section 3, we derive two new recursive schemes based on HPM for solving the doubly singular boundary value problems (1)–(3). Section 4 is devoted to convergence analysis of the methods. In Section 5, we present some numerical examples and compare numerical results obtained by the present method to those of some existing numerical methods. Finally in Section 6, we summarize and discuss the numerical results.

2. Homotopy-perturbation method

In this section, we will give a brief outline of the homotopy-perturbation method. For this, consider the nonlinear differential equation:

$$G(y(x)) + f(r(x)) = 0, \tag{4}$$